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The moduli space of flat $SU(2)$ and $SO(3)$ connections over surfaces

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Abstract

All the connected components of the moduli space of flat connections on $SU(2)$ and $SO(3)$ (trivial and non-trivial) bundles over closed oriented surfaces are determined. The symplectic structure and volumes of the non-maximal strata of the moduli space are also determined. © 1998 Elsevier Science B.V.

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1. Introduction

In this paper we shall study the moduli space $\mathcal{M}^0$ of flat connections on principal G-bundles over closed orientable surfaces, where G is $SU(2)$ or $SO(3)$.

Each moduli space is made up of several strata $\mathcal{M}_k^0$, each of which is a smooth k-dimensional manifold. In the case of $SO(3)$, the moduli space of flat connections on the trivial bundle is denoted $\mathcal{M}^0(I)$ (and the strata $\mathcal{M}_k^0(I)$), and the corresponding space for the non-trivial bundle is denoted $\mathcal{M}^0(-I)$ (and the strata $\mathcal{M}_k^0(-I)$). The detailed structure of the individual strata are described in Theorems 2.1, 3.1, 3.2, 3.7, 3.9, 3.20 and 3.24.

There is a standard symplectic structure on the infinite dimensional space of all connections over a closed oriented surface. It is known that this induces a symplectic structure on the maximal stratum of $\mathcal{M}^0$. In Section 6 we prove that *a symplectic structure is also induced on each of the lower-dimensional strata of* $\mathcal{M}^0$. The volume of the maximal stratum

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Table 1

Group/bundle	Stratum	Number of components	Volume
$SU(2)$trivial bundle	$\mathcal{M}^0_{3(2g-2)}$	$1(0 \text{ if } g=1)$	$2\mathrm{vol}\,(SU(2))^{2g-2} \times \sum_{n=1}^{\infty} \frac{1}{n^{2g-2}}$
	$\mathcal{M}^0_{2g}$	1	$\frac{1}{2}\,[4\pi \mathrm{vol}\,(SU(2))]^{2g/3}$
	$\mathcal{M}^0_0$	2^{2g}	
	$\mathcal{M}^0_{3(2g-2)}(I)$	$1(0 \text{ if} g=1)$	$2^{1-2g}\mathrm{vol}\,(SU(2))^{2g-2} \times \sum_{n=1}^{\infty} \frac{1}{n^{2g-2}}$
$SO(3)$trivial bundle	$\mathcal{M}^0_{2g}(I)$	1	$\frac{1}{2}\left[\frac{\pi \mathrm{vol}\,(SU(2))}{2}\right]^{2g/3}$
	$\mathcal{M}^0_{2g-2}(I)$	$2^{2g}-1(0 \text{ if } g=1)$	$\frac{1}{2}\left[\frac{\pi \mathrm{vol}\,(SU(2))}{2}\right]^{(2g-2)/3}$
	$\mathcal{M}^0_0(I)$	$\frac{1}{12}[2^{4g}+7\cdot 2^{2g}+4]$	
$SO(3)$ non-trivial bundle	$\mathcal{M}^0_{3(2g-2)}(-I)$	$1(0 \text{ if } g=1)$	$2^{1-2g}\mathrm{vol}\ (SU(2))^{2g-2} \times \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^{2g-2}}$
	$\mathcal{M}^0_{2g-2}(-I)$	$2^{2g}-1(0 \text{ if } g=1)$	$\frac{1}{2}\left[\frac{\pi \mathrm{vol}\,(SU(2))}{2}\right]^{(2g-2)/3}$
	$\mathcal{M}^0_0(-I)$	$\frac{1}{12}[16^g-4^g]$	

Note: $\mathcal{M}^0_k(z)$ is the stratum of dimension k.

of $\mathcal{M}^0$ has been determined in other works ([3,9], for instance). In Section 7 we *work out the volumes of the lower-dimensional strata* $\mathcal{M}^0_k(z)$, for $SU(2)$ and $SO(3)$.

Table 1 gives a summary of some of the results of this paper (the volumes of the maximal strata are not computed in the present work; see [9, (3.26,28),(4.73)]).

References to the literature on flat connections over surfaces may be found in [2,3,9,10].

2. The moduli space of flat $SU(2)$ connections

Let Σ be a compact connected oriented two-dimensional manifold of genus $g \geq 1$. As is well known, the moduli space $\mathcal{M}^0$ of flat $SU(2)$ connections over Σ may be identified with the quotient $K_g^{-1}(I)/SU(2)$, where K_g is the product commutator map

$$K_g : SU(2)^{2g} \to SU(2) : (a_1, b_1, \ldots, a_g, b_g) \mapsto b_g^{-1}a_g^{-1}b_g a_g \ldots b_1^{-1}a_1^{-1}b_1 a_1, \tag{2.1}$$

and $SU(2)$ acts on $K_g^{-1}(I)$ by conjugation in each component (Section 5 has some detail on this identification). In this section we shall use this identification of $\mathcal{M}^0$, along with its topology and smooth structure, with $K_g^{-1}(I)/SU(2)$. The main result of this section is:

Theorem 2.1. *The moduli space $\mathcal{M}^0$ is connected.*

Moreover, $\mathcal{M}^0$ is the union of disjoint sets $\mathcal{M}^0_{3(2g-2)}$, $\mathcal{M}^0_{2g}$ and $\mathcal{M}^0_0$, where:

(i) *$\mathcal{M}^0_{3(2g-2)}$ is empty if $g = 1$, while for $g \geq 2$ it is a smooth connected manifold of dimension $3(2g-2)$;*

(ii) *$\mathcal{M}^0_{2g}$ is a smooth connected $2g$-dimensional manifold, diffeomorphic to the quotient $(S^1)^{2g}\backslash\{\pm 1\}^{2g}/W$, where S^1 is the usual circle group of unit modulus complex numbers, and W is a two-element group $\{I, n\}$ acting on $(S^1)^{2g}$ by $n \cdot (z_1, \ldots, z_{2g}) = (z_1^{-1}, \ldots, z_{2g}^{-1})$;*

(iii) *$\mathcal{M}^0_0$ is a set consisting of 2^{2g} points.*

The proof of this will be completed by combining several results we shall prove below. However, we shall sketch first the general outline of the argument. The conjugation action of $SU(2)$ on $SU(2)^{2g}$ carries $K_g^{-1}(I)$ into itself and we may decompose $K_g^{-1}(I)$ according to the type of isotropy groups:

$$K_g^{-1}(I) = \mathcal{F}_{3(2g-2)} \cup \mathcal{F}_{2g} \cup \{\pm I\}^{2g}, \tag{2.2}$$

where

(i) $\mathcal{F}_{3(2g-2)}$ is the set of points where the isotropy group is $\{\pm I\}$, and

(ii) $\mathcal{F}_{2g}$ the set of points where the isotropy group is a torus in $SU(2)$.

We have then the corresponding decomposition

$$\mathcal{M}^0 = \mathcal{M}^0_{3(2g-2)} \cup \mathcal{M}^0_{2g} \cup \mathcal{M}^0_0, \tag{2.3}$$

where

$$\mathcal{M}^0_{3(2g-2)} = \mathcal{F}_{3(2g-2)}/SU(2) \quad \text{and} \quad \mathcal{M}^0_{2g} = \mathcal{F}_{2g}/SU(2), \tag{2.4}$$

The connectivity of $\mathcal{M}^0$ and the structures of the strata $\mathcal{M}^0_{3(2g-2)}$ and $\mathcal{M}^0_{2g}$ will be obtained by analyzing the sets $K_g^{-1}(I)$, $\mathcal{F}_{3(2g-2)}$, and $\mathcal{F}_{2g}$.

2.1. The isotropy groups

The following simple result (Section 3.7 in [11], Proposition B.III in [4]) is very useful:

Lemma 2.2. *Let H be a compact connected Lie group, equipped with an Ad-invariant metric. Consider the map*

$$\kappa_r : H^{2r} \to H : (x_1, y_1, \ldots, x_r, y_r) \mapsto y_r^{-1}x_r^{-1}y_rx_r \ldots y_1^{-1}x_1^{-1}y_1x_1,$$

and the conjugation action of H on H^{2r} given by (writing $x = (x_1, y_1, \ldots, x_r, y_r)$):

$$H \times H^{2r} \to H^{2r} : (a, x) \mapsto \gamma_x(a) = (ax_1a^{-1}, ay_1a^{-1}, \ldots, ax_ra^{-1}, ay_ra^{-1}).$$

For $x \in H$, let $Z(x)$ be the set of elements of H which commute with x. Thus the isotropy group $\mathcal{I}_x$ of the action of H at $x = (x_1, y_1, \ldots, x_r, y_r)$ is equal to $Z(x_1) \cap Z(y_1) \cap \cdots \cap Z(x_r) \cap Z(y_r)$. Then

$$\ker(\mathrm{d}\kappa_r|_x^*) = \text{Lie algebra of } \mathcal{I}_x = \ker \mathrm{d}\gamma_x|_e$$

(where e is the identity element of H).

The following describes the isotropy groups of the conjugation action of $SU(2)$ on $SU(2)^k$.

Lemma 2.3. *Let $x = (x_1, \ldots, x_k) \in SU(2)^k$. The isotropy group at x of the action of $SU(2)$ on $SU(2)^k$ is either $SU(2)$, or a maximal torus T, or $\{\pm I\}$:*

$$\text{the isotropy group} = \begin{cases} SU(2) & \text{if each } x_i \in \{\pm I\}; \\ \text{a maximal torus } T & \text{if all the } x_i, x_j \text{ commute with each other (thereby all lying in a maximal torus } T\text{) but are not all } \pm I; \\ \{\pm I\} & \text{if there exist two elements in } \{x_1, \ldots, x_k\} \text{ which do not commute.} \end{cases}$$

Proof. The case where the isotropy group is $SU(2)$ is clear. The other cases may be deduced from the following observations. If $a, b \in SU(2)$, $b \neq \pm I$, and $ab = ba$, then a belongs to the maximal torus containing b; this is readily verified by taking b to be a diagonal matrix. On the other hand, suppose $ab \neq ba$, and consider $x \in Z(a) \cap Z(b)$, $x \neq \pm I$; then, taking a to be diagonal, we see that, since $a \neq \pm I$, x is also diagonal and, since $x \neq \pm I$, this implies that b is diagonal, thus contradicting $ab \neq ba$. Thus $Z(a) \cap Z(b) = \{\pm I\}$ if $ab \neq ba$. □

2.2. The product commutator map

We list some useful observations about the product commutator map:

Lemma 2.4. *Let r be an integer ≥ 1, and consider the map*

$$K_r : SU(2)^{2r} \to SU(2) : (x_1, y_1, \ldots, x_r, y_r) \mapsto y_r^{-1} x_r^{-1} y_r x_r \ldots y_1^{-1} x_1^{-1} y_1 x_1.$$

(i) *The map K_r is surjective.*
(ii) *The critical points of K_r all lie in $K_r^{-1}(I)$.*
(iii) *$K_1^{-1}(I)$ is the set of critical points of K_1.*
(iv) *If $(x_1, y_1, \ldots, x_r, y_r)$ is a critical point of K_r then $Z(x_1) \cap Z(y_1) \cap \cdots \cap Z(x_r) \cap Z(y_r)$ is either $SU(2)$ or a maximal torus in $SU(2)$.*
(v) *If $(x_1, y_1, \ldots, x_r, y_r)$ is not a critical point of K_r then $Z(x_1) \cap Z(y_1) \cap \cdots \cap Z(x_r) \cap Z(y_r) = \{\pm I\}$.*
(vi) *$(x_1, y_1, \ldots, x_r, y_r)$ is a critical point of K_r if and only if $x_1, y_1, \ldots, x_r, y_r$ all lie in one maximal torus in $SU(2)$ (they commute with each other).*

Proof. (i) This is a general fact valid for compact connected topological groups having finite center, not only for $SU(2)$. But for $SU(2)$, it suffices to observe that any

$$\begin{pmatrix} \beta & 0 \\ 0 & \overline{\beta} \end{pmatrix} \in SU(2)$$

can be written as $b^{-1}a^{-1}ba$ for some $a, b \in SU(2)$; for instance,

$$b = \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \quad \text{and} \quad a = \begin{pmatrix} \alpha & 0 \\ 0 & \overline{\alpha} \end{pmatrix},$$

wherein α is a square-root of β.

(ii)–(vi) follow by combining Lemmas 2.2 and 2.3. For example, for (ii), if $x = (x_1, y_1, \ldots, x_r, y_r)$ is a critical point of K_r then, by Lemma 2.2, the isotropy group at x of the $SU(2)$ action on $SU(2)^{2r}$ has non-zero Lie algebra. Then, by Lemma 2.3, all the x_i, y_j commute, and so $K_r(x) = I$. □

2.3. *Decomposition of $K_r^{-1}(c)$ into manifolds*

If $c \in SU(2)\backslash\{I\}$ then, by Lemma 2.4(ii), c is a regular value of K_g and so $K_g^{-1}(c)$ is a smooth submanifold of $SU(2)^{2g}$. So we shall focus on $K_g^{-1}(I)$. As noted in (2.2), we have the decomposition

$$K_g^{-1}(I) = \mathcal{F}_{3(2g-2)} \cup \mathcal{F}_{2g} \cup \{\pm I\}^{2g} \tag{2.5a}$$

according to the isotropy type of the conjugation action of $SU(2)$ on $K_g^{-1}(I)$. Since $\mathcal{F}_{3(2g-2)}$ is, by definition, the set of all points on $K_g^{-1}(I)$ where the isotropy group of the $SU(2)$ conjugation action is $\{\pm I\}$, it follows from Lemmas 2.3 and 2.4(iv) and (v) that

$$\mathcal{F}_{3(2g-2)} = K_g^{-1}(I) \cap U_{\text{nc}}, \tag{2.5b}$$

where U_{nc} is the set of all non-critical points of K_g.

If $g = 1$ then, by Lemma 2.4(iii), $K_g^{-1}(I)$ consists only of the critical points of K_g and so, by (2.5b), $\mathcal{F}_{3(2g-2)}$ is empty.

Now suppose $g \geq 2$. Then, by the surjectivity of K_g (Lemma 2.4(i)), we can pick $x = (x_1, y_1, \ldots, x_g, y_g) \in K_g^{-1}(I)$ for which $K_1(x_1, y_1) \neq I$. Then, by Lemma 2.4(v), x is not a critical point of K_g. Thus $\mathcal{F}_{3(2g-2)}$ is non-empty, if $g \geq 2$. Thus, when $g \geq 2$,

$$\begin{aligned} \mathcal{F}_{3(2g-2)} = (K_g|U_{\text{nc}})^{-1}(I) &\text{ is a smooth } 3(2g-1)\text{-dimensional submanifold} \\ &\text{of (the open set } U_{\text{nc}} \subset) SU(2)^{2g}. \end{aligned} \tag{2.5c}$$

Next we consider $\mathcal{F}_{2g}$. By definition, $\mathcal{F}_{2g}$ consists of those points in $K_g^{-1}(I)$ where the isotropy group is a maximal torus in $SU(2)$. Let T be a maximal torus in $SU(2)$. Thus the map

$$\Phi^1 : SU(2) \times T^{2g} \to SU(2)^{2g} : (x, t_1, \ldots, t_{2g}) \mapsto (xt_1x^{-1}, \ldots, xt_{2g}x^{-1}) \tag{2.6a}$$

has image $\mathcal{F}_{2g} \cup \{\pm I\}^{2g}$; this follows from Lemma 2.3.

Computing $d\Phi^1$ at a point $(x, p) = (x, (t_j)_j)$, we have

$$d\Phi^1(xX, (t_j v_j)_j) = \Phi^1(x, p)\mathrm{Ad}(x)[v_j - (1 - \mathrm{Ad}(t_j^{-1})X]. \tag{2.6b}$$

Splitting X as $X_{||} + X_{\perp}$, where $X_{||} \in L(T)$ (the Lie algebra of T) and $X_{\perp} \in L(T)^{\perp}$, we see that $(xX, (t_j v_j)_j)$ lies in $\ker d\Phi^1$ if and only if each v_j is 0 and $\mathrm{Ad}(t_j)X_{\perp} = X_{\perp}$, for each j. If some $t_j \neq \pm I$ then the condition $\mathrm{Ad}(t_j)X_{\perp} = X_{\perp}$ is equivalent to $X_{\perp} = 0$, i.e. $X \in L(T)$. Thus the map Φ^1 induces, by restriction and quotient, an immersion

$$\Phi : (SU(2)/T) \times (T^{2g}\backslash\{\pm I\}^{2g}) \to SU(2)^{2g} \tag{2.6c}$$

whose image is $\mathcal{F}_{2g}$. Examining Φ, we see that it induces a continuous one-to-one map

$$\overline{\Phi} : [(SU(2)/T) \times (T^{2g}\backslash\{\pm I\}^{2g})]/W \to SU(2)^{2g} \tag{2.6d}$$

with image $\mathcal{F}_{2g}$, where the quotient $[\cdots]/W$ is under the action of $W = N(T)/T \simeq \{I, n\}$, the Weyl group of T, on $(SU(2)/T) \times T^{2g}$ specified by

$$nT \cdot (xT, t_1, \ldots, t_{2g}) = (xn^{-1}T, t_1^{-1}, \ldots, t_{2g}^{-1}).$$

This action is free and restricts to a free action on $(SU(2)/T) \times (T^{2g}\backslash\{\pm I\}^{2g})$, and so $[(SU(2)/T) \times (T^{2g}\backslash\{\pm I\}^{2g})]/W$ is a smooth manifold, the corresponding quotient map being a 2-fold covering. Since Φ^1 maps closed sets to closed sets, the map $\overline{\Phi}$ takes closed sets to (relatively) closed subsets of $\mathcal{F}_{2g}$; thus $\overline{\Phi}$ gives a homeomorphism onto $\mathcal{F}_{2g}$, taken as a subspace of $SU(2)^{2g}$. Since Φ is an immersion, so is $\overline{\Phi}$. Thus

$$\mathcal{F}_{2g} \text{ is a submanifold of } SU(2)^{2g}, \tag{2.7a}$$

and $\overline{\Phi}$ gives a diffeomorphism onto $\mathcal{F}_{2g}$. In particular,

$$\dim \mathcal{F}_{2g} = 2g + 2. \tag{2.7b}$$

Thus $K_g^{-1}(I)$ is the union of the disjoint sets $\mathcal{F}_{3(2g-2)}$, $\mathcal{F}_{2g}$, $\{\pm I\}^{2g}$, where $\mathcal{F}_{3(2g-2)}$ is a $3(2g-1)$-dimensional submanifold of $SU(2)^{2g}$ and $\mathcal{F}_{2g}$ is a $(2g+2)$-dimensional submanifold of $SU(2)^{2g}$.

Note that each of the manifolds making up $K_g^{-1}(I)$ is of codimension ≥ 2 in $SU(2)^{2g}$.

2.4. *Structure and connectivity of the sets $K_g^{-1}(c)$*

We will prove that each $K_g^{-1}(c)$ is connected and, furthermore, that the manifolds $\mathcal{F}_{3(2g-2)}$ and $\mathcal{F}_{2g}$ (which make up $K_g^{-1}(I)$) are also connected.

The arguments for connectivity of $K_g^{-1}(c)$ and $\mathcal{F}_{2g}$ will have a Morse theoretic flavor but we will work through detailed 'elementary' arguments, since these will yield additional facts which will be useful for other purposes.

The space $\mathcal{F}_{2g}$ is connected because it is the image of a connected space under the continuous map $\overline{\Phi}$, as seen in (2.6d).

We turn now to $K_g^{-1}(c)$. The argument will be inductive, with the following observation leading to the first inductive step.

Lemma 2.5. *Let $r \geq 1$ and let $C : SU(2)^{2r} \to SU(2)$ be a product of commutators of some of the pairs (x_i, y_i) (more precisely, $C = C_{i_1} \cdots C_{i_k}$ for some distinct $i_1, \ldots, i_k \in \{1, \ldots, r\}$). Then there is a diffeomorphism*

$$\psi : (SU(2)\backslash\{I\}) \times C^{-1}(-I) \to SU(2)^{2r}\backslash C^{-1}(I) \tag{2.8a}$$

such that the following diagram commutes:

$$\begin{array}{ccc} (SU(2)\backslash\{I\}) \times C^{-1}(-I) & \overset{\psi}{\to} & SU(2)^{2r}\backslash C^{-1}(I) \\ \searrow \mathrm{pr}_1 & & \swarrow C \\ & SU(2)\backslash\{I\} & \end{array} \tag{2.8b}$$

where pr_1 is the projection on the first factor.

Proof. If $p \in SU(2)^{2r}\backslash C^{-1}(I)$ then p is not a critical point of C (this follows from Lemma 2.4(iii)). Thus C is a submersion of $SU(2)^{2r}\backslash C^{-1}(I)$ onto $SU(2)\backslash\{I\}$. Moreover, C is a proper map. Then by Ehresmann's theorem [1, 20.8, prob. 4] C is a fibration. Since $SU(2)\backslash\{I\}$ is contractible, it follows that C is a trivial fiber bundle. □

Next we have our first connectivity result for $K_r^{-1}(c)$:

Proposition 2.6. *For any $h \in SU(2)\backslash\{I\}$, $K_1^{-1}(h)$ is a smooth manifold diffeomorphic to $SO(3)$. In particular, $K_1^{-1}(h)$ is connected for every $h \neq I$.*

Proof. In view of the preceding result, it will suffice to prove that $K_1^{-1}(-I)$ is diffeomorphic to $SO(3)$. Let

$$a_0 = \begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix} \quad \text{and} \quad b_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix};$$

then $b_0^{-1}a_0^{-1}b_0a_0 = -I$. It is proven in Lemma 3.13 of [6] that $\phi : SU(2)/\{\pm 1\} \mapsto SU(2)^2 : \pm x \mapsto (xa_0x^{-1}, xb_0x^{-1})$ maps $SU(2)/\{\pm I\}$ onto $K_1^{-1}(-I)$. Since a_0 and b_0 do not commute, Lemma 2.3 says that $Z(a_0) \cap Z(b_0) = \{\pm I\}$. Thus ϕ is one-to-one. The map ϕ is smooth, and its derivative is given by

$$\phi(x)^{-1}\mathrm{d}\phi|_x X = (\mathrm{Ad}\,(x)(\mathrm{Ad}\,(a_0^{-1}) - 1)X, \mathrm{Ad}\,(x)(\mathrm{Ad}\,(b_0^{-1}) - 1)X).$$

Thus any $X \in \ker \phi(x)^{-1}\mathrm{d}\phi|_x$ commutes with both a_0 and b_0; so, since a_0 and b_0 do not lie in any one maximal torus, it follows from Lemma 2.3 that X must be 0. Thus ϕ has no critical points. Since $-I$ is a regular value of K_1 (Lemma 2.4(ii)), it follows that $K_1^{-1}(-I)$ is a (compact) submanifold of $SU(2)^2$. We conclude that $\phi : SU(2)/\{\pm I\} \to K_1^{-1}(-I)$ is a diffeomorphism; since $SU(2)/\{\pm I\} \simeq SO(3)$, we see that $K_1^{-1}(-I)$ is diffeomorphic to $SO(3)$. □

Let C_k be the commutator in the pair (x_k, y_k) in $(x_1, y_1, \ldots, x_g, y_g)$, i.e.

$$C_k : SU(2)^{2g} \to SU(2) : (x_1, \ldots, y_g) \mapsto y_k^{-1} x_k^{-1} y_k x_k. \tag{2.9a}$$

Then $K_g = C_g \ldots C_1$, and so

$$K_g^{-1} \mathrm{d}K_g = \sum_{j=1}^{g} Ad(C_{j-1} \ldots C_1)^{-1} C_j^{-1} \mathrm{d}C_j, \tag{2.9b}$$

which implies that if some C_j is not critical at a point p then K_g is also not critical at p.

We will now prove the connectivity of $K_g^{-1}(h)$. The argument is inductive. The strategy is to focus on the subset $\mathcal{F}^1(h)$ of $K_g^{-1}(h)$ on which both C_1 and $C_g \cdots C_2$ are non-critical. As we will see, the 'projection' $C_1 : \mathcal{F}^1(h) \to SU(2)\backslash\{I, h\}$ is a surjective submersion and has connected compact fibers. This will imply that $\mathcal{F}^1(h)$ is connected. Next, connectivity of $K_g^{-1}(h)$ will be established by showing that any point in $K_g^{-1}(h)$ can be connected by a path to some point on $\mathcal{F}^1(h)$.

Proposition 2.7. *$K_r^{-1}(h)$ is connected, for every integer $r \geq 1$, and every $h \in SU(2)$. The set $\mathcal{F}^1(h)$, consisting of all points in $K_r^{-1}(h)$ where $C_1 \notin \{I, h\}$, is also connected (and non-empty when $r \geq 2$).*

Proof. We will write G for $SU(2)$. It has been shown in Proposition 2.6 that $K_1^{-1}(h)$ is connected when $h \neq I$. The connectedness of $K_1^{-1}(I)$ follows from the observation that, with T being a maximal torus in $SU(2)$, the map $G \times T^2 \to K_1^{-1}(I) : (x, a, b) \mapsto (xax^{-1}, xbx^{-1})$ is a continuous surjection (this follows from Lemma 2.4(iii) and (vi)).

Now let $N \geq 2$, and assume that $K_r^{-1}(c)$ is connected for every $c \in SU(2)$ and every $r = 1, \ldots, N-1$.

We will show first that $\mathcal{F}^1(h)$ is connected. The set $\mathcal{F}^1(h)$ consists of all points $x \in G^{2N}$ where $K_N(x) = h$ but $C_1(x) \notin \{I, h\}$, i.e.

$$\mathcal{F}^1(h) = C_1^{-1}(G\backslash\{I, h\}) \cap K_N^{-1}(h) \subset G^{2N}.$$

It follows from Lemma 2.4(i) that $\mathcal{F}^1(h) \neq \emptyset$. Moreover,

$$C_1(\mathcal{F}^1(h)) = G\backslash\{I, h\},$$

for if $g_1 \in G\backslash\{I, h\}$, then by Lemma 2.4(i), we can choose $p = (x_1, \ldots, y_N) \in G^{2N}$ such that $C_1(p) = g_1$ and $C_N(p) \cdots C_2(p) = hg_1^{-1}$, and thus $p \in \mathcal{F}^1(h)$.

Being a level set of K_N in an open subset of the set of non-critical points of C_1, $\mathcal{F}^1(h)$ is a smooth submanifold of G^{2N} (by (2.9b), K_N is not critical when C_1 is not critical). It follows from Lemma 4.1 (see Section 4 for a detailed explanation) that the map $C_1|\mathcal{F}^1(h) : \mathcal{F}^1(h) \to G$ is a submersion. If $z \in G\backslash\{I, h\}$ then the level set $(C_1|\mathcal{F}^1(h))^{-1}(z) = C_1^{-1}(z) \cap K_N^{-1}(h)$ is compact and connected, being (homeomorphic to) $K_1^{-1}(z) \times K_{N-1}^{-1}(hz^{-1})$, which is connected by the induction hypothesis on K_{N-1}. Thus $C_1|\mathcal{F}^1(h) : \mathcal{F}^1(h) \to G\backslash\{I, h\}$ is a surjective submersion with compact connected fibers $(C_1|\mathcal{F}^1(h))^{-1}(z)$. This implies that $\mathcal{F}^1(h)$ *is connected*: for if $p, q \in \mathcal{F}^1(h)$, then we can choose a path

$c : [0, 1] \to G\backslash\{I, h\}$ from $C_1(p)$ to $C_1(q)$ and then, by the submersive surjectivity of $C_1|\mathcal{F}^1(h)$ and compactness of the fibers of C_1, we can find a path $\tilde{c} : [0, 1] \to \mathcal{F}^1(h)$ with $\tilde{c}(0) = p$ and $\tilde{c}(1) \in (C_1|\mathcal{F}^1(h))^{-1}(C_1(q))$; connecting $\tilde{c}(1)$ to q by a path in $(C_1|\mathcal{F}^1(h))^{-1}(C_1(q))$ completes the argument.

To prove the connectivity of $K_N^{-1}(h)$ it will now suffice to show that any point in $K_N^{-1}(h)$ can be connected to a point in $\mathcal{F}^1(h)$ by a path lying in $K_N^{-1}(h)$. To this end let $p = (x_1, y_1, \dots, x_N, y_N) \in K_N^{-1}(h)\backslash\mathcal{F}^1(h)$; thus $C_1(p) \in \{I, h\}$.

Suppose $C_1(p) = h \neq I$. Then $K_{N-1}(x_2, y_2, \dots, x_N, y_N) = I$. Now, as we have seen earlier (2.5b) and (2.7a), $K_{N-1}^{-1}(I)$ is the union of at most three submanifolds of $G^{2(N-1)}$, each of positive codimension. So the point $(x_2, y_2, \dots, x_N, y_N)$ in the $6(N-1)$-dimensional manifold $K_{N-1}^{-1}(G\backslash\{h\})$ has an open connected neighborhood in which $K_{N-1}^{-1}(I)$ is the union of at most three positive-codimension submanifolds. Thus there is a path $[0, 1] \to G^{2(N-1)} : t \mapsto \tilde{p}_t$ such that : $\tilde{p}_0 = (x_2, y_2, \dots, x_N, y_N)$, $K_{N-1}(\tilde{p}_t) \neq h$ for all $t \in [0, 1]$ and $K_{N-1}(\tilde{p}_1) \neq I$. Thus $K_{N-1}(\tilde{p}_t)^{-1}h \neq I$ for all $t \in [0, 1]$ and $K_{N-1}(\tilde{p}_1)^{-1}h \neq h$. Then, since $K_1 : K_1^{-1}(G\backslash\{I\}) \to G\backslash\{I\}$ is a submersion with compact connected fibers $K_1^{-1}(z)$, it follows that there is a path $[0, 1] \to G^2 : t \mapsto p'_t$ with $p'_0 = (x_1, y_1)$ and $K_1(p'_t) = K_{N-1}(\tilde{p}_t)^{-1}h$. Then $p_t \stackrel{\text{def}}{=} (p'_t, \tilde{p}_t) \in K_N^{-1}(h)$, $p_0 = p$, and $p_1 \in \mathcal{F}^1(h)$. Thus we have connected the point p to a point in $\mathcal{F}^1(h)$ by a path in $K_N^{-1}(h)$.

Now suppose $C_1(p) = I \neq h$. We wish to show that there is a path in $K_N^{-1}(h)$ from p to $\mathcal{F}^1(h)$. Since $K_1^{-1}(I)$ is connected, we may assume that

$$y_1 = \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \quad \text{and} \quad x_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Let

$$x_1(t) = \begin{pmatrix} \mathrm{e}^{\mathrm{i}t} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}t} \end{pmatrix} \quad \text{and} \quad y_1(t) = y_1.$$

Then the path $[0, 1] \to G^2 : t \mapsto c(t) = (x_1(t), y_1(t))$, starts $(x_1(0), y_1(0)) = (x_1, y_1)$, and $K_1(c(t)) = x_1(2t) \notin \{I, h\}$ for t near 0 but $t \neq 0$. At $t = 0$ we have $K_1(c(0)) = C_1(p) = I$. Since $K_N(p) = I$ and $C_1(p) = I \neq h$, we have $C_N(p) \cdots C_2(p) = h \neq I$. So, by Lemma 2.4(vi), $K_{N-1} : G^{2(N-1)} \to G$ is a submersion in a neighborhood of $p' = (x_2, y_2, \dots, x_N, y_N)$. Then by our usual argument there is a path $c_{N-1} : [0, 1] \to G^{2(N-1)}$ such that $c_{N-1}(0) = p'$ and, for t near 0,

$$K_{N-1}(c_{N-1}(t)) = hK_1(c(t))^{-1}.$$

Thus $K_N(c(t), c_{N-1}(t)) = h$, and $(c(t), c_{N-1}(t)) \in \mathcal{F}^1(h)$ for small $t \neq 0$. Thus, if $h \neq I$, we have connected p to a point in $\mathcal{F}^1(h)$ by a path in $K_N^{-1}(h)$.

Finally, suppose $C_1(p) = I$ and $h = I$. Since $K_1^{-1}(I)$ and (by the inductive hypothesis) $K_{N-1}^{-1}(I)$ are connected, so is $C_1^{-1}(I) \cap K_N^{-1}(I) \simeq K_1^{-1}(I) \times K_{N-1}^{-1}(I)$. So we can connect the point $p \in C_1^{-1}(I) \cap K_N^{-1}(I)$ to the point $(I, b, \dots, I, b) \in C_1^{-1}(I) \cap K_N^{-1}(I)$, wherein

$$b = \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix},$$

by a path lying in $C_1^{-1}(I) \cap K_N^{-1}(I)$. So it will suffice to connect the point $(I, b, \ldots, I, b)$ to a point in $\mathcal{F}^1(I)$ by a path in $K_N^{-1}(I)$. Now let

$$x_1(t) = \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix} \quad \text{and} \quad y_1(t) = b;$$

then a simple calculation shows that $K_1(x_1(t), y_1(t)) = x_1(2t)$. Therefore,

$$K_N(x_1(t'), y_1(t'), \ldots, x_1(t'), y_1(t'), x_1(t), y_1(t)) = I,$$

where $t' = -t/(N-1)$.

Thus

$$t \mapsto p(t) = \left(x_1\left(-\frac{t}{N-1}\right), y_1\left(-\frac{t}{N-1}\right), \ldots, x_1\left(-\frac{t}{N-1}\right), y_1\left(-\frac{t}{N-1}\right), x_1(t), y_1(t)\right)$$

is a path in $K_N^{-1}(I)$, which for $t \neq 0$, but near 0, lies on $\mathcal{F}^1(I)$. Of course, $p(0)$ is $(I, b, \ldots, I, b)$, the chosen starting point. Thus $p(0)$ is connectable to a point in $\mathcal{F}^1(h)$ by a path in $K_N^{-1}(h)$. □

Finally, we prove that $\mathcal{F}_{3(2g-2)}$ is connected. This will be done by showing that $\mathcal{F}^1(I)$ is a dense subset of $\mathcal{F}_{3(2g-2)}$; since $\mathcal{F}^1(I)$ is connected, it will follow that so is $\mathcal{F}_{3(2g-2)}$. The density of $\mathcal{F}^1(I)$ will be proved by showing that the complement $C_1^{-1}(I) \cap \mathcal{F}_{3(2g-2)}$ is contained in a finite union of submanifolds of $\mathcal{F}_{3(2g-2)}$ each of codimension ≥ 1. The reason why $C_1^{-1}(I) \cap \mathcal{F}_{3(2g-2)}$ is easier to understand is that it is an open subset of $C_1^{-1}(I) \cap K_g^{-1}(I) = K_1^{-1}(I) \times K_{g-1}^{-1}(I)$, where the first factor can be understood in explicit terms while the second factor can be handled by induction.

Proposition 2.8. *Let $g \geq 2$, and recall that $\mathcal{F}_{3(2g-2)}$ is the set of points in $K_g^{-1}(I)$ where the isotropy group of the conjugation action of $SU(2)$ is $\{\pm I\}$. Then the set $\mathcal{F}^1(I)$, consisting of all points $(x_1, y_1, \ldots, x_g, y_g)$ in $\mathcal{F}_{3(2g-2)}$ with commutator $y_1^{-1}x_1^{-1}y_1x_1 \neq I$, is dense in $\mathcal{F}_{3(2g-2)}$. Consequently, $\mathcal{F}_{3(2g-2)}$ is connected.*

Proof. Let $G = SU(2)$, and $C_1 : G^{2g} \to G$ the commutator in the first pair (x_1, y_1). Then the complement of $\mathcal{F}^1(I)$ in $K_g^{-1}(I)$ is $C_1^{-1}(I) \cap K_g^{-1}(I) = K_1^{-1}(I) \times K_{g-1}^{-1}(I)$. Recall from (2.5a) and (2.7b) that $K_1^{-1}(I)$ is the union of $\{\pm I\}^2$ and a four-dimensional manifold, and, for $r > 1$, $K_r^{-1}(I)$ is the union of three submanifolds of $SU(2)^{2r}$ each of dimension $< 3(2r-1)$.

Thus if $g = 2$ then $C_1^{-1}(I) \cap K_g^{-1}(I)$ is the union of the four submanifolds of $SU(2)^4$, each of dimension ≤ 8. Recall that, for $g = 2$, $\mathcal{F}_{3(2g-2)}$ has dimension $3(2.2-1) = 9$ and is the intesection of $K_g^{-1}(I)$ with the open set U_{nc} of all non-critical points of K_g. Thus, intersecting with U_{nc}, we see that for $g = 2$, $C_1^{-1}(I) \cap \mathcal{F}_{3(2g-2)}$ is the union of four submanifolds of $\mathcal{F}_{3(2g-2)}$, each of codimension ≥ 1. Therefore, the complement $\mathcal{F}^1(I)$ is, in this case, dense in $\mathcal{F}_{3(2g-2)}$.

Now suppose $g > 2$. Then $K_{g-1}^{-1}(I)$ is the union of three submanifolds of $G^{2(g-1)}$ each of dimension $\leq 3(2(g-1)-1)$. So $C_1^{-1}(I) \cap K_g^{-1}(I)$ is the union of six submanifolds of $SU(2)^{2g}$ each of dimension $\leq 3(2(g-1)-1)+4 = 6g-5$. Since $\dim \mathcal{F}_{3(2g-2)} = 6g-3$, we see that $C_1^{-1}(I) \cap \mathcal{F}_{3(2g-2)}$ is the union of a finite number of submanifolds of $\mathcal{F}_{3(2g-2)}$ each of codimension ≥ 2. Hence, the complement $\mathcal{F}^1(I)$ is dense in $\mathcal{F}_{3(2g-2)}$. □

2.5. *Bundle structures over the strata of $\mathcal{M}^0$*

We have shown that $K_g^{-1}(I)$ is the union of disjoint sets $\mathcal{F}_{3(2g-2)}$, $\mathcal{F}_{2g}$, and $\{\pm I\}^{2g}$, where $\mathcal{F}_{3(2g-2)}$ and $\mathcal{F}_{2g}$ are submanifolds of $SU(2)^{2g}$. The moduli space $\mathcal{M}^0$ is identifiable with the quotient $K_g^{-1}(I)/SU(2)$. Thus we should understand the quotients $\mathcal{F}_{3(2g-2)} \to \mathcal{F}_{3(2g-2)}/SU(2)$ and $\mathcal{F}_{2g} \to \mathcal{F}_{2g}/SU(2)$.

Proposition 2.9. *For $g \geq 2$, the quotient space $\mathcal{F}_{3(2g-2)}/SU(2)$ is a manifold of dimension $3(2g-2)$, and the quotient map $\mathcal{F}_{3(2g-2)} \to \mathcal{F}_{3(2g-2)}/SU(2)$ is a principal $SO(3)$-bundle.*

Proof. We have already seen that $\mathcal{F}_{3(2g-2)}$ is a smooth $3(2g-1)$-dimensional submanifold of $SU(2)^{2g}$, the conjugation action of $SU(2)$ on $\mathcal{F}_{3(2g-2)}$ is smooth, being the restriction of the action on $SU(2)^{2g}$, and, by definition of $\mathcal{F}_{3(2g-2)}$, has isotropy group $\{\pm I\}$ everywhere. Therefore, the quotient space $\mathcal{F}_{3(2g-2)}/SU(2)$ is a smooth $3(2g-2)$-dimensional manifold and the quotient map $\mathcal{F}_{3(2g-2)} \to \mathcal{F}_{3(2g-2)}/SU(2)$ is a principal $SU(2)/\{\pm I\}$-bundle (see Proposition 4.2). To conclude, we use the fact that $SU(2)/\{\pm I\} \simeq SO(3)$. □

Next we shall show that $\mathcal{F}_{2g} \to \mathcal{F}_{2g}/SU(2)$ is a fiber bundle and identify it with a specific bundle over $\mathcal{F}_{2g}/SU(2)$. Let T be a maximal torus in $SU(2)$, and $W = \{I, n\}$ the corresponding Weyl group acting on T by $n(t) = ntn^{-1} = t^{-1}$. Then, as noted after (2.7a), $\mathcal{F}_{2g}$ can be identified with $[(SU(2)/T) \times (T^{2g}\backslash\{\pm I\}^{2g})]/W$.

The quotient projection $(T^{2g}\backslash\{\pm I\}^{2g}) \to (T^{2g}\backslash\{\pm I\}^{2g})/W$ is a principal W-bundle (i.e. a 2-fold covering). The group $W = \{I, n\}$ has a right action on $SU(2)/T$ in the usual way, with n acting by $xT \mapsto xn^{-1}T$. Thus we have a corresponding fiber bundle, with fiber $SU(2)/T$, associated to the principal W-bundle $(T^{2g}\backslash\{\pm I\}^{2g}) \to (T^{2g}\backslash\{\pm I\}^{2g})/W$.

Proposition 2.10. *The quotient space $\mathcal{F}_{2g}/SU(2)$ is a manifold and the quotient map $\mathcal{F}_{2g} \to \mathcal{F}_{2g}/SU(2)$ is a smooth fiber bundle isomorphic (in the smooth category) to the fiber bundle with fiber $SU(2)/T$ associated to the principal W-bundle (or covering) $(T^{2g}\backslash\{\pm I\}^{2g}) \to (T^{2g}\backslash\{\pm I\}^{2g})/W$, where $W = \{I, n\}$ acts on $SU(2)/T$ by $xT \mapsto xT$ and $xT \mapsto xn^{-1}T$.*

Proof. As we have seen before in the context of (2.6a), the map (with $G = SU(2)$)

$$\Phi^1 : (G/T) \times T^{2g} \to G^{2g} : (xT, t_1, \ldots, t_{2g}) \mapsto (xt_1x^{-1}, \ldots, xt_{2g}x^{-1}) \qquad (2.10a)$$

has image $\mathcal{F}_{2g} \cup \{\pm I\}^{2g}$, and induces by restriction and quotient a continuous one-to-one map

$$\overline{\Phi} : [(G/T) \times (T^{2g}\backslash\{\pm I\}^{2g})]/W \to G^{2g} \tag{2.10b}$$

with image $\mathcal{F}_{2g}$, where the quotient $[\cdots]/W$ is under the right action of W specified by $(n \in W, n \neq I)$

$$nT \cdot (T, t_1, \ldots, t_{2g}) = (xn^{-1}T, t_1^{-1}, \ldots, t_{2g}^{-1}).$$

This action is free and restricts to a free action on $(G/T) \times \left(T^{2g}\backslash\{\pm I\}^{2g}\right)$, and so the quotient $[(G/T) \times (T^{2g}\backslash\{\pm I\}^{2g})]/W$ is a smooth manifold, the corresponding quotient map being a 2-fold covering. As seen in (2.7b), $\mathcal{F}_{2g}$ is a submanifold of G^{2g} and $\overline{\Phi}$ gives a diffeomorphism onto $\mathcal{F}_{2g}$.

The natural left action of G on G/T gives a left action of G on $(G/T) \times T^{2g}$ (which commutes with the right action of W), and a corresponding action on the quotient space $[(G/T) \times (T^{2g}\backslash\{\pm I\}^{2g})]/W$. It is readily verified that $\overline{\Phi}$ is G-equivariant.

These considerations may be illustrated by the commuting diagram :

$$\begin{array}{ccc} [(SU(2)/T) \times (T^{2g}\backslash\{\pm I\}^{2g})]/W & \overset{\overline{\Phi}}{\to} & \mathcal{F}_{2g} \\ \downarrow p & & \downarrow p' \\ [T^{2g}\backslash\{\pm I\}^{2g}]/W & \overset{\overline{\overline{\Phi}}}{\to} & \mathcal{F}_{2g}/SU(2) \end{array} \tag{2.10c}$$

where p is obtained from the projection of $(SU(2)/T) \times T^{2g}$ on the second factor, p' is the quotient map, and $\overline{\overline{\Phi}}$ is the induced map. Clearly $\overline{\overline{\Phi}}$ is a homeomorphism.

We observe that p is a smooth fiber bundle projection: it is the G/T-bundle associated to the principal W-bundle $T^{2g}\backslash\{\pm I\}^{2g} \to (T^{2g}\backslash\{\pm I\}^{2g})/W$ by the action of W on G/T (specified by $n \cdot xT \mapsto xn^{-1}T$). As already noted, $\overline{\Phi}$ is a diffeomorphism and $\overline{\overline{\Phi}}$ is a homeomorphism. Thus the projection $\mathcal{F}_{2g} \overset{p'}{\to} \mathcal{F}_{2g}/G$ is a submersion if and only if $\mathcal{F}_{2g}/G$ is equipped with the smooth structure which makes $\overline{\overline{\Phi}}$ a diffeomorphism; and with this smooth structure, the projection $\mathcal{F}_{2g} \to \mathcal{F}_{2g}/G$ is a smooth fiber bundle with fiber G/T and structure group W, isomorphic (in the smooth category) to the bundle given by p. □

Proof of Theorem 2.1 We can now put together all the pieces to obtain Theorem 2.1.

Recall that the moduli space $\mathcal{M}^0$ of flat connections over the compact oriented genus $g(\geq 1)$ surface Σ is identified with the quotient space $K_g^{-1}(I)/SU(2)$. Then $\mathcal{M}^0$ is the disjoint union $\mathcal{M}^0_{3(2g-2)} \cup \mathcal{M}^0_{2g} \cup \mathcal{M}^0_0$, where $\mathcal{M}^0_{3(2g-2)}$ corresponds to the quotient $\mathcal{F}_{3(2g-2)}/SU(2)$, the stratum $\mathcal{M}^0_{2g}$ corresponds to $\mathcal{F}_{2g}/SU(2)$, and $\mathcal{M}^0_0$ is a set of 2^{2g} points corresponding to $\{\pm I\}^{2g}/SU(2)$. We have already proved that $\mathcal{F}_{3(2g-2)}$ is empty when $g = 1$, while for $g \geq 2$ it is a connected $3(2g-2)$-dimensional manifold. We have also proved, in Proposition 2.10, that $\mathcal{F}_g/SU(2)$ is a connected $2g$-dimensional manifold, as given in (2.10c). □

3. The moduli spaces of flat $SO(3)$ connections

Let Σ be a compact connected oriented two-dimensional manifold of genus $g \geq 1$. Then there are two topologically distinct classes of principal $SO(3)$-bundles over Σ, one trivial and the other non-trivial. The moduli space of flat connections on the trivial bundle will be denoted $\mathcal{M}^0(I)$, and the moduli space of flat connections on the non-trivial bundle will be denoted $\mathcal{M}^0(-I)$. The main results are:

Theorem 3.1. *The moduli space $\mathcal{M}^0(I)$ is the union of disjoint subsets*

$$\mathcal{M}^0(I) = \mathcal{M}^0_{3(2g-2)}(I) \cup \mathcal{M}^0_{2g}(I) \cup \mathcal{M}^0_{2g-2}(I) \cup \mathcal{M}^0_0(I), \tag{3.1}$$

where

(i) $\mathcal{M}^0_{3(2g-2)}(I)$ *is a connected* $3(2g-2)$*-dimensional manifold (empty if and only if* $g = 1$*),*
(ii) $\mathcal{M}^0_{2g}(I)$ *is a connected* $2g$*-dimensional manifold,*
(iii) $\mathcal{M}^0_{2g-2}(I)$ *is empty if* $g = 1$*, while for* $g \geq 2$ *it is a* $(2g-2)$*-dimensional manifold with* $2^{2g} - 1$ *components,*
(iv) $\mathcal{M}^0_0(I)$ *is a finite set.*

For the non-trivial bundle the corresponding result is:

Theorem 3.2. *The moduli space $\mathcal{M}^0(-I)$ is the union of disjoint subsets:*

$$\mathcal{M}^0(-I) = \mathcal{M}^0_{3(2g-2)}(-I) \cup \mathcal{M}^0_{2g-2}(-I) \cup \mathcal{M}^0_0(-I), \tag{3.2}$$

where

(i) $\mathcal{M}^0_{3(2g-2)}(-I)$ *is a connected* $3(2g-2)$*-dimensional manifold (empty if and only if* $g = 1$*),*
(ii) $\mathcal{M}^0_{2g-2}(-I)$ *is a* $(2g-2)$*-dimensional manifold with* $2^{2g} - 1$ *components (empty if and only if* $g = 1$*),*
(iii) $\mathcal{M}^0_0(-I)$ *is a finite set.*

In this section we shall often write G for $SU(2)$, and $\overline{G}$ for $SO(3)$. There is a standard covering map $G \to SO(3) : x \mapsto \overline{x}$, whose kernel is $\{\pm I\}$. If $\overline{y} \in SO(3)$, we will denoted by y any element in $SU(2)$ which covers $\overline{y}$.

The product commutator map

$$\tilde{K}_g : SO(3)^{2g} \to G : (\overline{a}_1, \overline{b}_1, \ldots, \overline{a}_g, \overline{b}_g) \mapsto b_g^{-1} a_g^{-1} b_g a_g \ldots b_1^{-1} a_1^{-1} b_1 a_1 \tag{3.3}$$

will be useful. Since the kernel of the covering map $G \to SO(3)$ is (in) the center of G, $\tilde{K}_g$ is well-defined.

The moduli space $\mathcal{M}^0(I)$ of flat connections on the trivial bundle can be identified with quotient $\tilde{K}_g^{-1}(I)/SO(3)$, while the moduli space $\mathcal{M}^0(-I)$ of flat connections on the non-trivial bundle can be identified with $\tilde{K}_g^{-1}(-I)/SO(3)$:

$$\mathcal{M}^0(I) \simeq \tilde{K}_g^{-1}(I)/SO(3) \quad \text{and} \quad \mathcal{M}^0(-I) \simeq \tilde{K}_g^{-1}(-I)/SO(3). \tag{3.4}$$

The strategy is again to understand the structure of $\mathcal{M}^0(z) \simeq \tilde{K}_g^{-1}(z)/SO(3)$ by separating out the subsets of $\tilde{K}_g^{-1}(z)$ corresponding to different isotropy groups of the $SO(3)$ action.

We are using the following decomposition:

$$\tilde{K}_g^{-1}(z) = \overline{\mathcal{F}}_{3(2g-2)}(z) \cup \overline{\mathcal{F}}_{2g}(z) \cup \overline{\mathcal{F}}_{2g-2}(z) \cup \overline{\mathcal{F}}_0(z), \tag{3.5a}$$

where $z = \pm I$, and

(i) $\overline{\mathcal{F}}_{3(2g-2)}(z)$ is the subset of $\tilde{K}_g^{-1}(z)$ where the isotropy of the $SO(3)$-action is $\{I\}$,

(ii) $\overline{\mathcal{F}}_{2g}(z)$ is the subset where the isotropy group is a maximal torus in $SO(3)$,

(iii) $\overline{\mathcal{F}}_{2g-2}(z)$ is the subset where the isotropy group consists of two elements (the identity and a 180° rotation),

(iv) $\overline{\mathcal{F}}_0(z)$ is the remaining subset of $\tilde{K}_g^{-1}(z)$; as we shall see in Proposition 3.4 below, the only other possible isotropy groups are: (a) $SO(3)$, (b) the normalizer $N(K)$ of a maximal torus K of $SO(3)$, (c) a four-element group $\{I, n_1, n_2, n_3\}$, where $\{n_1, n_2, n_3\}$ are 180° rotations around orthogonal axes.

(The set $\overline{\mathcal{F}}_0(z)$ should not be confused with $\overline{\mathcal{F}}_{2g-2}(z)$ or with $\overline{\mathcal{F}}_{3(2g-2)}(z)$ for $g = 1$.)

Then we decompose the moduli space as

$$\mathcal{M}^0(z) = \mathcal{M}^0_{3(2g-2)}(z) \cup \mathcal{M}^0_{2g}(z) \cup \mathcal{M}^0_{2g-2}(z) \cup \mathcal{M}^0_0(z). \tag{3.5b}$$

where $\mathcal{M}^0_{3(2g-2)}(z)$ is the subset corresponding to $\overline{\mathcal{F}}_{3(2g-2)}(z)/SO(3)$, and similarly for $\mathcal{M}^0_{2g}(z)$, $\mathcal{M}^0_{2g-2}(z)$, and $\mathcal{M}^0_0(z)$.

3.1. The isotropy groups of the $SO(3)$-action

We start with a few preliminary observations. Some of these may be verified by taking the covering map $SU(2) \to SO(3)$ to be given by means of the adjoint representation of $SU(2)$ on its Lie algebra $\underline{g}$; the vector space $\underline{g}$ can be identified with $\mathbf{R}^3$ using a basis which is orthonormal with respect to an Ad-invariant metric on $\underline{g}$.

Observations 3.3.

(i) A maximal torus in $SO(3)$ corresponds to rotations around a fixed axis in $\mathbf{R}^3$.

(ii) Elements $a, b \in SO(3)$ satisfy $\tilde{b}^{-1}\tilde{a}\tilde{b} = -\tilde{a}$, where $\tilde{a}, \tilde{b} \in SU(2)$ cover $a, b \in SO(3)$, if and only if a and b are 180° rotations around orthogonal axes (this may be verified by considering a diagonal form for $\tilde{a}$, for instance). Thus an element $a \in SO(3)$ commutes with $b \in SO(3)$ if and only if either a and b lie in the same maximal torus or they are 180° rotations around orthogonal axes.

(iii) Let $a \in SO(3)$, $\overline{T}$ a maximal torus in $SO(3)$ and suppose $aba^{-1} \in \overline{T}$ for some $b \in \overline{T}\backslash\{I\}$. Considering covering elements $\tilde{a}, \tilde{b} \in SU(2)$, with $\tilde{b}$ taken diagonal by suitably conjugating $\overline{T}$, it follows by matrix computation that $a \in N(\overline{T})$ (the normalizer of $\overline{T}$) and $aba^{-1} = b^{\pm 1}$. Conversely, if $a \in N(\overline{T})\backslash\overline{T}$ and $b \in \overline{T}$ then $aba^{-1} = b^{-1}$; this may also be verified by passing to $SU(2)$.

(iv) By (iii) and (ii), $N(\overline{T})\backslash\overline{T}$ consists of all the 180° rotations about axes orthogonal to the axis for $\overline{T}$.

Proposition 3.4. *Let $H_x \subset SO(3)$ be the isotropy group at a point $x = (x_1, \ldots, x_r) \in SO(3)^r$ of the conjugation action of $SO(3)$ on $SO(3)^r$, $r \geq 1$.*

(i) *$H_x = SO(3)$ if and only if $x = (I, \ldots, I)$, i.e. $\{x_1, \ldots, x_r\} = \{I\}$.*

(ii) *$H_x = N(K) = K \cup nK$, the normalizer of a maximal torus K in $SO(3)$ (thus $n \in N(K)\backslash K$), if and only if $\{x_1, \ldots, x_r\} \subset \{I, \tau\}$ for some 180° rotation τ (the 180° rotation belonging to $\overline{T}$) and $\{x_1, \ldots, x_r\} \neq \{I\}$.*

(iii) *$H_x = \{I, n_1, n_2, n_3\}$, where n_1, n_2, n_3 are 180° rotations around three orthogonal axes, if and only if: $\{n_1, n_2\} \subset \{x_1, \ldots, x_r\} \subset \{I, n_1, n_2, n_3\}$ (i.e. $\{x_1, \ldots, x_r\} \subset \{I, n_1, n_2, n_3\}$ but there is no 180° rotation τ such that $\{x_1, \ldots, x_r\} \subset \{I, \tau\}$).*

(iv) *$H_x = K$, a maximal torus in $SO(3)$, if and only if $x_1, \ldots, x_r \in K$ and there is no 180° rotation τ such that $\{x_1, \ldots, x_r\} \subset \{I, \tau\}$.*

(v) *$H_x = \{I, \tau\}$, for some 180° rotation n, if and only if: there is a maximal torus K (containing τ) and 180° rotations $n_1, \ldots, n_j$, with axes orthogonal to that for K, such that $\{x_1, \ldots, x_r\} \subset K \cup \{n_1, \ldots, n_j\}$ (i.e., $\{x_1, \ldots, x_r\} \subset N(K)$) but x does not satisfy the conditions of* (i)–(iv) *above.*

(vi) *$H_x = \{I\}$ if and only if the conditions of* (i)–(v) *do not hold, i.e. there is no maximal torus K such that $\{x_1, \ldots, x_r\} \subset N(K)$.*

Proof.

(i) Apparent.

(ii) Suppose $\{I\} \neq \{x_1, \ldots, x_r\} \subset \{I, \tau\}$, for some 180° rotation τ. Then $H_x = \{y \in SO(3) : y\tau y^{-1} = \tau\}$; by Observations 3.3 (ii) and (iv), this set equals $N(K)$, the normalizer of the maximal torus K containing τ. Conversely, suppose $H_x = N(K)$. Then each x_i commutes with every element of K, and so each x_i must $\in K$. Moreover, choosing $n \in N(K)\backslash K$, we have $x_i = nx_in^{-1} = x_i^{-1}$, and so $x_i^2 = I$. Since $H_x \neq SO(3)$, x cannot be $(I, \ldots, I)$; thus $x = (x_1, \ldots, x_r)$, with $\{I\} \neq \{x_1, \ldots, x_r\} \subset \{I, \tau\}$.

(iv) is proved by arguments similar to those used for (ii).

(iii) Suppose that there are 180° rotations n_1, n_2 and n_3, around orthogonal axes, such that $\{n_1, n_2\} \subset \{x_1, \ldots, x_r\} \subset \{I, n_1, n_2, n_3\}$. If $y \in H_x$ then y commutes with n_1 and n_2 and hence, by Observation 3.3(ii), must belong to $\{I, n_1, n_2, n_3\}$. It also follows from Observation 3.3(ii) that $\{I, n_1, n_2, n_3\} \subset H_x$; thus $H_x = \{I, n_1, n_2, n_3\}$. Conversely, suppose $H_x = \{I, n_1, n_2, n_3\}$, the n_i's being 180° rotations around orthogonal axes. Then, by Observation 3.3(ii), each x_i must either be in $\{I, n_1, n_2, n_3\}$ or be a 180° rotation with axis orthogonal to those of n_1, n_2 and n_3. The latter being impossible, we conclude that $\{x_1, \ldots, x_r\} \subset \{I, n_1, n_2, n_3\}$. Now if $\{x_1, \ldots, x_r\}$ were a subset of $\{I, n_1\}$ then H_x would, by (i) and (ii), not be equal to $\{I, n_1, n_2, n_3\}$. Thus H_x must contain at least two 180° rotations; taking these to be n_1 and n_2, we conclude that $\{n_1, n_2\} \subset \{x_1, \ldots, x_r\} \subset \{I, n_1, n_2, n_3\}$.

(v) Suppose $H_x = \{I, \tau\}$, where τ is a 180° rotation. Since, by Observations 3.3, the set of elements which commute with τ equals $N(K)$, the normalizer of the maximal torus K containing τ, it follows that $\{x_1, \ldots, x_r\} \subset N(K)$; since H_x contains two elements, the conditions for (i)–(iv) cannot hold.

Conversely, suppose that $\{x_1, \ldots, x_r\} \subset N(K)$, where $N(K)$ is the normalizer of a maximal torus K, and the conditions for (i)–(iv) do not hold. Then $\{I, \tau\} \subset H_x$ because τ commutes with every element of $N(K)$. Since (i)–(iii) do not apply, there is at least one $x_j \in N(K) \backslash K$. If there is only one $x_j \in N(K) \backslash K$ then, since (ii) and (iv) do not apply, there is some $i \in \{1, \ldots, r\}$ with $x_i \in K$ and $x_i^2 \neq I$; in this case $H_x \subset Z(x_i) \cap Z(x_j) = \{I, \tau\}$, and so $H_x = \{I, \tau\}$. Now suppose there exist distinct $x_j, x_k \in N(K) \backslash K$. If x_j and x_k have orthogonal axes then, since (ii) and (iv) do not apply, there is some $x_i \in K$ with $x_i^2 \neq I$ and so, as before, $H_x = \{I, \tau\}$. Finally, if $x_j, x_k \in N(K) \backslash K$ have non-orthogonal axes then $H_x \subset Z(x_j) \cap Z(x_k) = \{I, \tau\}$, and so again $H_x = \{I, \tau\}$.

(vi) Suppose $\{x_1, \ldots, x_r\} \subset N(K)$ for some maximal torus K. Then, by Observation 3.3(ii) and (iv), the 180° rotation $\tau \in K$ commutes with each x_i and so H_x cannot be $\{I\}$. Conversely, if $H_x \neq \{I\}$ then, choosing $h \in H_x \backslash \{I\}$, and letting K be the maximal torus containing h, Observation 3.3 shows that $N(K)$ is the set of all elements of $SO(3)$ which commute with h, and so $\{x_1, \ldots, x_r\} \subset N(K)$. □

3.2. *The structure of $\overline{\mathcal{F}}_{3(2g-2)}(\pm I)$*

Recall that $\overline{\mathcal{F}}_{3(2g-2)}(z)$ is the set of all points in $\tilde{K}_g^{-1}(z)$ where the isotropy of the $SO(3)$-action is $\{I\}$.

Proposition 3.5. *If $g \geq 2$ then $\overline{\mathcal{F}}_{3(2g-2)}(I)$ is non-empty and is a* connected $3(2g-1)$-*dimensional submanifold of $SO(3)^{2g}$. If $g = 1$ then $\overline{\mathcal{F}}_{3(2g-2)}(I)$ is empty.*

Proof. Recall that $\mathcal{F}_{3(2g-2)}$, the subset of $K_g^{-1}(I) \subset SU(2)^{2g}$ where the conjugation action of $SU(2)$ has isotropy group $\{\pm I\}$, is the part of the level set $K_g^{-1}(I)$ which lies in the set of non-critical points of K_g. If $\overline{p} \in \overline{\mathcal{F}}_{3(2g-2)}(I)$ then, by Lemma 2.2, $\tilde{K}_g$ is not critical at $\overline{p}$ and so, since the covering $SU(2) \to SO(3)$ is a local diffeomorphism, K_g is not critical at p, and therefore $p \in \mathcal{F}_{3(2g-2)}$. Thus $\overline{\mathcal{F}}_{3(2g-2)}(I)$ is a subset of $\overline{\mathcal{F}}_{3(2g-2)}$, the projection of $\mathcal{F}_{3(2g-2)}$ on $SO(3)^{2g}$. If $g = 1$ then $\mathcal{F}_{3(2g-2)} = \emptyset$ and hence so is $\overline{\mathcal{F}}_{3(2g-2)}(I)$.

We proceed with the case $g \geq 2$.

Pick $a, b \in SU(2)$ such that: (i) a, b do not commute, (ii) $a^2, b^2 \notin \{\pm I\}$; for example:

$$a = \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix},$$

where $t = \pi/4$. By Lemma 2.4(i), we can choose $c, d \in SU(2)$ satisfying $d^{-1}c^{-1}dc = (b^{-1}a^{-1}ba)^{-1}$. Then, recalling that $g \geq 2$, we have $(\overline{a}, \overline{b}, \overline{c}, \overline{d}, I, I, \ldots, I) \in \tilde{K}_g^{-1}(I)$ and $Z(\overline{a}) \cap Z(\overline{b}) \cap Z(\overline{c}) \cap Z(\overline{d}) = \{I\}$; for if $x \in SU(2)$ satisfies $xax^{-1} = \pm a$ and $xbx^{-1} = \pm b$ then, since $a^2 \neq \pm I$ and $b^2 \neq \pm I$, it follows (by Observation 3.3(ii)) that $xax^{-1} = a$ and $xbx^{-1} = b$, and thus, since $b^{-1}a^{-1}ba \neq I$, x must be $\pm I$, and so $\overline{x} = I (\in SO(3))$. Thus, $(\overline{a}, \overline{b}, \overline{c}, \overline{d}, I, I, \ldots, I) \in \overline{\mathcal{F}}_{3(2g-2)}(I)$. So, if $g \geq 2$ then $\overline{\mathcal{F}}_{3(2g-2)}(I) \neq \emptyset$.

Let $\mathcal{W}$ be the set of points of $SO(3)^{2g}$ at which the isotropy group of the $SO(3)$ conjugation action is $\{I\}$. It is readily seen that $\mathcal{W}$ is non-empty. Let us check that it is open. Consider a sequence $p_1, p_2, \ldots$ of points in $\mathcal{W}^c$ converging to some $p \in SO(3)^{2g}$. From Proposition 3.4 we see that for any $q \in SO(3)^{2g}$, the isotropy group H_q is either $\{I\}$ or contains a 180° rotation. Thus each isotropy group H_{p_j} contains a 180° rotation x_j. After passing to a subsequence if necessary, we take x_j converging to a point x, and have

$$xpx^{-1} = \lim_{j\to\infty} x_j p_j x_j^{-1} = \lim_{j\to\infty} p_j = p,$$

i.e. $x \in H_p$. Since each x_j is a 180° rotation, so is x. Thus the limit point p does not belong to $\mathcal{W}$. Thus $\mathcal{W}$ is open. In fact, the complement of $\mathcal{W}$, being the subset of $SO(3)^{2g}$ covered by Proposition 3.4(i)–(iv), consists of the union of a finite number of submanifolds of dimension $\leq 2g + 3$ and so is $\mathcal{W}$ a dense open subset of $SO(3)^{2g}$. (Actually, a general result in the theory of transformation groups implies that $\mathcal{W}$ is a dense open subset of $SO(3)^{2g}$.) By Lemma 2.2, $\tilde{K}_g$ has no critical points in $\mathcal{W}$; therefore, $\overline{\mathcal{F}}_{3(2g-2)}(I)$, being the level set $(\tilde{K}_g|\mathcal{W})^{-1}(I)$, and being non-empty if $g \geq 2$, is, in that case, a $3(2g - 1)$-dimensional submanifold of $SO(3)^{2g}$.

As we have already noted, $\overline{\mathcal{F}}_{3(2g-2)}(I) \subset \overline{\mathcal{F}}_{3(2g-2)}$. Thus $\overline{\mathcal{F}}_{3(2g-2)}(I)$ is the subset of $\overline{\mathcal{F}}_{3(2g-2)}$ consisting of the points where the $SO(3)$-conjugation-action is free. Let U'_{nc} be the subset of $SO(3)^{2g}$ consisting of all non-critical points of $\tilde{K}_g$; then U'_{nc} is open and $\overline{\mathcal{F}}_{3(2g-2)} = (\tilde{K}_g|U'_{\mathrm{nc}})^{-1}(I)$. Thus, for $g \geq 2$, $\overline{\mathcal{F}}_{3(2g-2)}$ is a smooth $3(2g - 1)$-dimensional submanifold of $SO(3)^{2g}$. Since $\mathcal{F}_{3(2g-2)}$ is connected, so is its continuous image $\overline{\mathcal{F}}_{3(2g-2)}$. The conjugation action of $SO(3)$ on $SO(3)^{2g}$ restricts to a smooth action on the invariant submanifold $\overline{\mathcal{F}}_{3(2g-2)}$. Since $\tilde{K}_g$ is non-critical at each point of $\overline{\mathcal{F}}_{3(2g-2)}$, it follows from Lemma 2.2 that the isotropy group at every point in $\overline{\mathcal{F}}_{3(2g-2)}$ is discrete. By Proposition 3.4, we know that this discrete isotropy group is either $\{I\}$, or a two-element group or a four-element group. As will be proven later in Propositions 3.13 and 3.22, the subset of $\overline{\mathcal{F}}_{3(2g-2)}$ consisting of points where the isotropy group is a two-element group or a four-element group is the union of a finite number of submanifolds each of dimension $\leq 2g+2$. Since these manifolds have codimension $\geq 4g - 5$, and since $\overline{\mathcal{F}}_{3(2g-2)}$ is connected, it follows that, for $g \geq 2$, $\overline{\mathcal{F}}_{3(2g-2)}(I)$ is connected. □

A general result in the theory of transformation groups says that the set of points of minimal isotropy is a dense open subset of the connected manifold on which the group acts, and the corresponding projection onto the quotient space is connected. In our setting, this also implies that $\overline{\mathcal{F}}_{3(2g-2)}(I)/SO(3)$ is connected.

Proposition 3.6. *If $g \geq 2$ then $\overline{\mathcal{F}}_{3(2g-2)}(-I)$ is non-empty and is a smooth* connected *manifold of dimension* $3(2g-1)$. *If $g = 1$ then $\overline{\mathcal{F}}_{3(2g-2)}(-I)$ is empty.*

Proof. If $g = 1$, and $(a, b) \in \tilde{K}_g^{-1}(-I)$, then, by Observation 3.3(ii), a and b are 180° rotations around orthogonal axes. In this case, the isotropy group at (a, b) is, according to Proposition 3.4(iii), a four-element group. Thus at no point on $\tilde{K}_1^{-1}(-I)$ does $SO(3)$ act freely, i.e. $\overline{\mathcal{F}}_{3(2g-2)}(-I)$ is empty if $g = 1$.

Now suppose $g \geq 2$. Pick $a, b \in SU(2)$ such that: (i) a, b do not commute, (ii) a^2 and b^2 are not in $\{\pm I\}$. Pick (by Lemma 2.4(i)) $c, d \in SU(2)$ such that $d^{-1}c^{-1}dc = -(b^{-1}a^{-1}ba)^{-1}$. Then $(\overline{a}, \overline{b}, \overline{c}, \overline{d}, \ldots) \in \tilde{K}_g^{-1}(-I)$ and, as in the proof of Proposition 3.5, the isotropy group at $(\overline{a}, \overline{b}, \overline{c}, \overline{d}, I, I, \ldots, I)$ is $\{I\}$. Thus $(\overline{a}, \overline{b}, \overline{c}, \overline{d}, I, I, \ldots, I) \in \overline{\mathcal{F}}_{3(2g-2)}(-I)$.

We work with $g \geq 2$. By Lemmas 2.4(ii) and 2.2, $-I$ is a regular value of $\tilde{K}_g$, and so $\tilde{K}_g^{-1}(-I)$ is a smooth $3(2g-1)$-dimensional submanifold of $SO(3)^{2g}$. As in the proof of Proposition 3.5, $\overline{\mathcal{F}}_{3(2g-2)}(-I)$ is an open subset of $\tilde{K}_g^{-1}(-I)$ and so is a $3(2g-1)$-dimensional submanifold of $SO(3)^{2g}$.

From Proposition 2.7, the manifold $K_g^{-1}(-I)$ is connected, and hence so is the projection $\tilde{K}_g^{-1}(-I)$. It will be proven in (3.6) and Proposition 3.22 that the subset of $\tilde{K}_g^{-1}(-I)$ consisting of all points where the $SO(3)$-conjugation action is not free is the union of a finite number of submanifolds each of dimension $\leq 2g+1$, i.e. of codimension $\geq 4g-4 \geq 4$ in $\tilde{K}_g^{-1}(-I)$. Thus the subset of $\tilde{K}_g^{-1}(-I)$ where the $SO(3)$-action is free is connected, i.e. $\overline{\mathcal{F}}_{3(2g-2)}(-I)$ is connected. □

We turn to the quotients.

Theorem 3.7. *Suppose $g \geq 2$, and $z = \pm I$. Then $\overline{\mathcal{F}}_{3(2g-2)}(z)/SO(3)$ is a connected smooth manifold of dimension* $3(2g-2)$, *and the projection map*

$$\overline{\mathcal{F}}_{3(2g-2)}(z) \to \overline{\mathcal{F}}_{3(2g-2)}(z)/SO(3)$$

is a smooth principal $SO(3)$-bundle.

Proof. Since $SO(3)$ acts freely on $\overline{\mathcal{F}}_{3(2g-2)}(z)$, the result follows from the general fact quoted in Proposition 4.2, and the connectivity proved in Propositions 3.5 and 3.6. □

3.3. *The structure of $\overline{\mathcal{F}}_{2g}(\pm I)$*

Recall that $\overline{\mathcal{F}}_{2g}(z)$ is the subset of $\tilde{K}_g^{-1}(z)$ where the isotropy group of the $SO(3)$-action is a maximal torus in $SO(3)$. According to Proposition 3.4 (iv) if a point $p = (a_1, b_1, \ldots, a_g, b_g) \in \overline{\mathcal{F}}_{2g}(z)$ then , there are covering elements $\tilde{a}_j$ and $\tilde{b}_j$ all lying in one maximal torus in $SU(2)$, and so $\tilde{K}_g(p) = I$. Thus

$$\overline{\mathcal{F}}_{2g}(-I) = \emptyset. \tag{3.6}$$

Proposition 3.8. *$\overline{\mathcal{F}}_{2g}(I)$ is a connected smooth submanifold of $SO(3)^{2g}$ of dimension $2g+2$.*

Proof. By definition, $\overline{\mathcal{F}}_{2g}(I)$ consists of those points in $\tilde{K}_g^{-1}(I)$ where the isotropy group is a maximal torus in $SO(3)$. Let $\overline{T}$ be a maximal torus in $SO(3)$, and τ the 180° rotation belonging to $\overline{T}$. For notational brevity, let us write $\overline{G}$ for $SO(3)$. Consider the map

$$(\overline{G}/\overline{T}) \times \overline{T}^{2g} \to SO(3)^{2g} : (x\overline{T}, t_1, \dots, t_{2g}) \mapsto (xt_1x^{-1}, \dots, xt_{2g}x^{-1}). \tag{3.7a}$$

By Proposition 3.4(iv), the restriction

$$\begin{aligned}\Phi_{SO(3)} &: (\overline{G}/\overline{T}) \times (\overline{T}^{2g} \backslash \{I, \tau\}^{2g}) \\ &\to SO(3)^{2g} : (x\overline{T}, t_1, \dots, t_{2g}) \mapsto (xt_1x^{-1}, \dots, xt_{2g}x^{-1})\end{aligned} \tag{3.7b}$$

has image $\overline{\mathcal{F}}_{2g}(I)$ (see the argument preceding (3.6)). It is readily verified (as in (2.6b)) by computation of the derivative $d\Phi_{\overline{G}}$, that $\Phi_{SO(3)}$ is an immersion.

Let W be the Weyl group of $\overline{T}$, i.e. $W = N(\overline{T})/\overline{T} \simeq \{I, n\}$, where n is a 180° rotation around an axis orthogonal to the axis for $\overline{T}$ (this follows from Observation 3.3). Examining $\Phi_{SO(3)}$, we see that it induces a continuous one-to-one map

$$\overline{\Phi}_{SO(3)} : [(\overline{G}/\overline{T}) \times (\overline{T}^{2g} \backslash \{I, \tau\}^{2g})]/W \to SO(3)^{2g}, \tag{3.7c}$$

where the quotient $[\cdots]/W$ is under the action of W on $(SO(3)/\overline{T}) \times \overline{T}^{2g}$ specified by

$$n\overline{T} \cdot (x\overline{T}, t_1, \dots, t_{2g}) = (xn^{-1}T, t_1^{-1}, \dots, t_{2g}^{-1}).$$

This action is free and restricts to a free action on $(SO(3)/\overline{T}) \times (\overline{T}^{2g} \backslash \{I, \tau\}^{2g})$, and so the quotient $[(SO(3)/\overline{T}) \times (\overline{T}^{2g} \backslash \{I, \tau\}^{2g})]/W$ is a smooth manifold, the corresponding quotient map being a 2-fold covering. The image of $\overline{\Phi}_{SO(3)}$ is $\overline{\mathcal{F}}_{2g}(I)$.

Since the map in (3.7a) takes closed sets to closed sets, the map $\overline{\Phi}_{SO(3)}$ takes closed sets to (relatively) closed subsets of $\overline{\mathcal{F}}_{2g}(I)$. Thus $\overline{\Phi}_{SO(3)}$ gives a homeomorphism onto $\overline{\mathcal{F}}_{2g}(I)$, taken as a subspace of $SO(3)^{2g}$. Since $\Phi_{SO(3)}$ is an immersion, so is $\overline{\Phi}_{SO(3)}$. Thus

$$\overline{\mathcal{F}}_{2g}(I) \text{ is a submanifold of } SO(3)^{2g}, \tag{3.8a}$$

and $\overline{\Phi}_{SO(3)}$ gives a diffeomorphism onto $\overline{\mathcal{F}}_{2g}(I)$. In particular,

$$\dim \overline{\mathcal{F}}_{2g}(I) = 2g+2. \tag{3.8b}$$

Theorem 3.9. *The quotient space $\overline{\mathcal{F}}_{2g}(I)/SO(3)$ is a connected smooth manifold of dimension $2g$. The quotient map $\overline{\mathcal{F}}_{2g}(I) \to \overline{\mathcal{F}}_{2g}(I)/SO(3)$ specifies a smooth fiber bundle isomorphic to a fiber bundle with fiber the sphere S^2 associated to a principal W-bundle over $\overline{\mathcal{F}}_{2g}(I)/SO(3)$, where W is the two-element group acting on S^2 by $x \mapsto -x$.*

Proof. As we have seen above, the map

$$(SO(3)/\overline{T}) \times \overline{T}^{2g} \to SO(3)^{2g} : (xT, t_1, \ldots, t_{2g}) \qquad \square$$
$$\mapsto (xt_1x^{-1}, \ldots, xt_{2g}x^{-1}) \tag{3.9a}$$

induces by restriction and quotient a diffeomorphism

$$\overline{\Phi} : [(SO(3)/\overline{T}) \times (\overline{T}^{2g} \backslash \{I, \tau\}^{2g})]/W \to \overline{\mathcal{F}}_{2g}(I), \tag{3.9b}$$

where the quotient $[\cdots]/W$ is under the right action of W specified by $(n \in W, n \neq I)$

$$nT \cdot (xT, t_1, \ldots, t_{2g}) = (xn^{-1}T, t_1^{-1}, \ldots, t_{2g}^{-1}). \tag{3.9c}$$

The natural left action of $\overline{G}$ on $SO(3)/\overline{T}$ gives a left action of $SO(3)$ on $(SO(3)/\overline{T}) \times \overline{T}^{2g}$ (which commutes with the right action of W), and a corresponding action on the quotient space $[(SO(3)/\overline{T}) \times (\overline{T}^{2g} \backslash \{I, \tau\}^{2g})]/W$. It is readily verified that $\overline{\Phi}$ is $SO(3)$-equivariant. We have then the commuting diagram

$$\begin{array}{ccc} [(SO(3)/\overline{T}) \times (\overline{T}^{2g} \backslash \{I, \tau\}^{2g})]/W & \stackrel{\overline{\Phi}}{\to} & \overline{\mathcal{F}}_{2g}(I) \\ \downarrow p & & \downarrow p' \\ [\overline{T}^{2g} \backslash \{I, \tau\}^{2g}]/W & \stackrel{\overline{\overline{\Phi}}}{\to} & \overline{\mathcal{F}}_{2g}(I)/SO(3) \end{array} \tag{3.9d}$$

where p is obtained from the projection of $(SO(3)/\overline{T}) \times \overline{T}^{2g}$ on the second factor, p' is the quotient map, and $\overline{\overline{\Phi}}$ is the induced map. The induced map $\overline{\overline{\Phi}}$ is one-to-one, and is therefore a homeomorphism.

We observe that p is a smooth fiber bundle projection: it is the $SO(3)/\overline{T}$-bundle associated to the principal W-bundle $\overline{T}^{2g} \backslash \{I, \tau\}^{2g} \to (\overline{T}^{2g} \backslash \{\pm I\}^{2g})/W$ by the action of W on $SO(3)/\overline{T}$ (specified by $n \cdot x\overline{T} \mapsto xn^{-1}\overline{T}$). As already noted, $\overline{\Phi}$ is a diffeomorphism and $\overline{\overline{\Phi}}$ is a homeomorphism. Thus the projection $\overline{\mathcal{F}}_{2g}(I) \stackrel{p'}{\to} \overline{\mathcal{F}}_{2g}(I)/SO(3)$ is a submersion if and only if $\overline{\mathcal{F}}_{2g}(I)/SO(3)$ is equipped with the smooth structure which makes $\overline{\overline{\Phi}}$ a diffeomorphism; and with this smooth structure, the projection $\overline{\mathcal{F}}_{2g}(I) \to \overline{\mathcal{F}}_{2g}(I)/SO(3)$ is a smooth fiber bundle with fiber $SO(3)/\overline{T} \simeq S^2$ and structure group W, isomorphic (in the smooth category) to the bundle given by p. □

3.4. The set of points in $SO(3)^{2g}$ where the isotropy has two elements

We have

$$\mathcal{M}^0_{2g-2}(z) \stackrel{\text{def}}{=} \overline{\mathcal{F}}_{2g-2}(z)/SO(3),$$

where $\overline{\mathcal{F}}_{2g-2}(z)$ is the set of all points in $\tilde{K}_g^{-1}(z)$ where the isotropy group of the $SO(3)$-conjugation action is a two-element group.

Suppose $g = 1$. Then, by Observation 3.3(ii), if $(a, b) \in \tilde{K}_g^{-1}(\pm I)$ then either a and b lie in the same maximal torus or they are 180° rotations around orthogonal axes. In either case, the isotropy group is not a two-element group (this by Proposition 3.4(i)–(iv)). Thus $\overline{\mathcal{F}}_{2g-2}(\pm I)$ is empty if $g = 1$.

We shall work now with $g \geq 2$.

Our immediate objective is to understand the subset of $SO(3)^{2g}$ consisting of points where the isotropy group has two elements.

Proposition 3.10. *Let*

$$F \stackrel{\text{def}}{=} \begin{cases} \textit{the subset of } SO(3)^{2g} \textit{consisting of all points} \\ \textit{where the isotropy group has two elements.} \end{cases} \tag{3.10}$$

Then

(a) *F is a $(2g+2)$-dimensional submanifold of $SO(3)^{2g}$.*

(b) *The quotient map $F \to F/SO(3)$ has the structure of a fiber bundle, with fiber $SO(3)/\{I, \tau\}$, where τ is a 180° rotation, and structure group $N(\overline{T})/\{I, \tau\}$, where $N(\overline{T})$ is the normalizer of the maximal torus $\overline{T}$ containing τ.*

We will break up the proof of this result into a number of lemmas.

We work with a fixed maximal torus $\overline{T}$ in $SO(3)$. Let τ be the 180° rotation belonging to $\overline{T}$, and fix any $n \in N(\overline{T})\backslash\overline{T}$, i.e. n is a 180° rotation with axis perpendicular to that of $\overline{T}$.

The conjugation action $SO(3) \times SO(3)^{2g} \to SO(3)^{2g}$ induces, by restriction, a smooth map

$$\Psi : SO(3) \times N(\overline{T})^{2g} \to SO(3)^{2g} : (x, p) \mapsto xpx^{-1}. \tag{3.11a}$$

We are interested in this map because Proposition 3.4(v) guarantees that the image of Ψ contains the subset of $SO(3)^{2g}$ where the isotropy group has two elements.

The map Ψ is invariant under the following action of $N(\overline{T})$ on $SO(3) \times N(\overline{T})^{2g}$:

$$y \cdot (x, p) \mapsto (xy^{-1}, ypy^{-1}), \quad \text{for } y \in N(\overline{T}). \tag{3.11b}$$

Let B denote the subset of $N(\overline{T})^{2g}$ consisting of all points where the isotropy group is not a two-element group. Proposition 3.4 yields the following explicit description of the set B:

$$B = \overline{T}^{2g} \cup B', \tag{3.11c}$$

where

$$B' = \left\{ \begin{matrix} (x_j) \in N(\overline{T})^{2g} : \textit{if } x_j \in \overline{T} \textit{ then } x_j \in \{I, \tau\}; \textit{if } x_j \in N(\overline{T})\backslash\overline{T} \textit{ then} \\ x_j \in \{yn, y\tau n\} \textit{ for some } y \in \overline{T} \textit{ independent of } j \end{matrix} \right\} \tag{3.11d}$$

The set B' is clearly contained in the union of $\{I, \tau\}^{2g}$ with a finite number of diffeomorphic images of $\overline{T}$. So B is a closed subset of $N(\overline{T})^{2g}$. Thus, $N(\overline{T})^{2g}\backslash B$ is a $2g$-dimensional manifold, with $2^{2g} - 1$ components.

Lemma 3.11. *Two points in $SO(3) \times [N(\overline{T})^{2g}\backslash B]$ are on the same $N(\overline{T})$-orbit if and only if they have the same image under Ψ.*

Proof. Since Ψ is invariant under the action of $N(\overline{T})$, the 'only if' part is clear.

For the 'if' part, suppose $\Psi(x, p) = \Psi(y, q)$, where $x, y \in N(\overline{T})^{2g}\backslash B$; i.e.

$$xpx^{-1} = yqy^{-1}.$$

Then

$$wpw^{-1} = q,$$

where $w = y^{-1}x$. It will suffice to show that w is in $N(\overline{T})$.

If some component p_j of p belongs to $\overline{T}\backslash\{1, \tau\}$, then $wp_jw^{-1} = q_j \in N(\overline{T})$ but since $(wp_jw^{-1})^2 \neq I$ (otherwise p_j would be τ), wp_jw^{-1} must be in $\overline{T}$ and so, by Observation 3.3(iii), $w \in N(\overline{T})$ (and therefore, $q_j = p_j^{\pm 1} \in \overline{T}$). The same argument works if $q_j \in \overline{T}\backslash\{1, \tau\}$.

So suppose now that if either p_j or q_j is in $\overline{T}$ then $p_j, q_j \in \{I, \tau\}$ (i.e. either $p_j, q_j \in N(\overline{T})\backslash\overline{T}$ or $p_j, q_j \in \{I, \tau\}$). Now consider a component $p_{j_1} \in N(\overline{T})\backslash\overline{T}$. By conjugating p by an appropriate element of $\overline{T}$ (and multiplying x, or w, on the right by that element), we will assume that $p_{j_1} = n$. Consider another component $p_{j_2} \in N(\overline{T})\backslash\overline{T}$, $p_{j_2} \neq p_{j_1}$. Since $wp_{j_1}w^{-1} = q_{j_1} \in N(\overline{T})\backslash\overline{T}$, we have $wnw^{-1} = tn$, $t \in \overline{T}$. Next, $wp_{j_2}w^{-1} = q_{j_2}$ implies $wsnw^{-1} = rn$, for some $s \in \overline{T}\backslash\{I\}$ and $r \in \overline{T}$. So $rn = wsnw^{-1} = wsw^{-1}tn$, and so $wsw^{-1} = rt^{-1} \in \overline{T}$. Hence $w \in N(\overline{T})$. □

The action of $N(\overline{T})$ on $SO(3) \times N(\overline{T})^{2g}$ is free and so the quotient is a smooth manifold and Ψ induces a smooth map

$$[SO(3) \times N(\overline{T})^{2g}]/N(\overline{T}) \to SO(3)^{2g}. \tag{3.12a}$$

Let $\overline{\Psi}$ denote the restriction of the map (3.12a) to the subset $SO(3) \times [N(\overline{T})^{2g}\backslash B]/N(\overline{T})$. According to Lemma 3.11, the map $\overline{\Psi}$ is one-to-one.

Lemma 3.12. *The map*

$$\overline{\Psi} : [SO(3) \times (N(\overline{T})^{2g}\backslash B)]/N(\overline{T}) \to SO(3)^{2g}$$

is an immersion.

Proof. Let $(x, p) \in SO(3) \times N(\overline{T})^{2g}$, and X be a vector in the Lie algebra of $SO(3)$, and $P \in L(\overline{T})^{2g}$. Thus (xX, pP) is a typical element of $T_{(x,p)}[SO(3) \times N(\overline{T})^{2g}]$. Recall that $\Psi(x, p) = xpx^{-1}$. Writing $P = (P_j)_j$, we have

$$\mathrm{d}\Psi(xX, pP) = xpx^{-1}(\mathrm{Ad}(x)[P_j - (1 - \mathrm{Ad}(p_j^{-1}))X])_j. \tag{3.12b}$$

Suppose (xX, pP) is in the kernel of $\mathrm{d}\Psi$. Write $X = X_{||} + X_{\perp}$, where $X_{||} \in L(\overline{T})$ and $X_{\perp} \in L(\overline{T})^{\perp}$ (this is the orthogonal complement relative to any Ad-invariant metric on the Lie algebra of $SO(3)$). Then, from (3.12b), we have, for each j,

$$(1 - \mathrm{Ad}p_j^{-1})X_{\perp} = 0, \tag{$*$}$$

$$(1 - \mathrm{Ad}p_j^{-1})X_{||} = P_j. \tag{$**$}$$

From (∗) it follows that $\exp(\epsilon X_\perp)$ commutes with p_j, for every real ϵ. Since $p \notin B$, the isotropy group at p has only two elements and therefore $X_\perp = 0$. Then, using (∗∗), we have

$$
\begin{aligned}
(xX, pP) &= \frac{\mathrm{d}}{\mathrm{d}\epsilon}\Big|_{\epsilon=0} (x\exp(\epsilon X), \exp(-\epsilon X)p\exp(\epsilon X)) \\
&= \frac{\mathrm{d}}{\mathrm{d}\epsilon}\Big|_{\epsilon=0} \exp(-\epsilon X)\cdot(x, p)
\end{aligned}
$$

Thus we have proved that if (xX, pP) is in the kernel of $\mathrm{d}\Psi$ then (xX, pP) is tangent to the $N(\overline{T})$–orbit through (x, p). □

Combining the above results, we see that the image of $\overline{\Psi}$ is a submanifold of $SO(3)^{2g}$ and $\overline{\Psi}$ is a diffeomorphism onto its image. This image is the union of all $SO(3)$ orbits through the points of $N(\overline{T})^{2g}$ where the isotropy group has two elements. Thus this image consists only of points where the isotropy group has two elements. Moreover, by Proposition 3.4(v), any point in $SO(3)^{2g}$ where the isotropy group has two elements is on the $SO(3)$-orbit through some point in $N(\overline{T})^{2g}$. Thus

$$\overline{\Psi}([SO(3)\times(N(\overline{T})^{2g}\backslash B)]/N(\overline{T})) = F.$$

As noted after (3.11d), the space $\big(N(\overline{T})^{2g}\backslash B\big)$ is a smooth $2g$-dimensional submanifold of $SO(3)^{2g}$, with $2^{2g}-1$ components. The quotient $[SO(3)\times(N(\overline{T})^{2g}\backslash B)]/N(\overline{T})$, being the quotient under a free action, is a smooth $(3+2g-1)$-dimensional manifold, and the corresponding quotient map is a principal $N(\overline{T})$-bundle projection map. Thus F is a $(2g+2)$-dimensional submanifold of $SO(3)^{2g}$. The $N(\overline{T})$-conjugation carries each component of $N(\overline{T})^{2g}$ into itself. Thus F also has $2^{2g}-1$ components.

We have proved Proposition 3.10(a) and more:

Proposition 3.13. *The set F of all points in $SO(3)^{2g}$ where the isotropy group has two elements is a smooth $(2g+2)$-dimensional submanifold of $SO(3)^{2g}$. Moreover,*

$$\overline{\Psi} : [SO(3)\times(N(\overline{T})^{2g}\backslash B)]/N(\overline{T}) \to F \text{ is a diffeomorphism.} \tag{3.13}$$

The group $SO(3)$ acts on $SO(3)\times\big(N(\overline{T})^{2g}\backslash B\big)$ by left-multiplication on the first factor, and this action commutes with the action of $N(\overline{T})$. Thus we have an induced natural action of $SO(3)$ on $[SO(3)\times\big(N(\overline{T})^{2g}\backslash B\big)]/N(\overline{T})$. The corresponding quotient is

$$[SO(3)\times(N(\overline{T})^{2g}\backslash B)]/N(\overline{T}) \overset{p}{\to} (N(\overline{T})^{2g}\backslash B)/N(\overline{T}), \tag{3.14a}$$

which is essentially the projection on the 'second factor'.

Clearly, $\overline{\Psi}$ is equivariant under the action of $SO(3)$. We have then the commutative diagram

$$
\begin{array}{ccc}
[SO(3)\times(N(\overline{T})^{2g}\backslash B)]/N(\overline{T}) & \overset{\overline{\Psi}}{\to} & \mathrm{Im}\,(\overline{\Psi}) = F \\
\downarrow p & & \downarrow p' \\
[N(\overline{T})^{2g}\backslash B]/N(\overline{T}) & \overset{\overline{\overline{\Psi}}}{\to} & \mathrm{Im}\,\overline{\Psi}/SO(3) = F/SO(3)
\end{array} \tag{3.14b}
$$

in which the quotient $[N(\overline{T})^{2g}\backslash B]/N(\overline{T})$ is with respect to the conjugation action, and the bottom arrow is induced by the inclusion $N(\overline{T})^{2g}\backslash B \to F \subset SO(3)^{2g}$.

Lemma 3.14. *The bottom arrow* $\overline{\overline{\Psi}}$ *in* (3.14b) *is a homeomorphism.*

Proof. Since $\overline{\Psi}$ is a homeomorphism and p and p' are quotient maps, it will suffice to prove that $\overline{\overline{\Psi}}$ is one-to-one. Injectivity of $\overline{\overline{\Psi}}$ is equivalent to $\overline{\Psi}$ mapping distinct $SO(3)$-orbits into distinct orbits. To this end, let $(x, s), (y, u) \in SO(3) \times N(\overline{T})^{2g}$ be such that there is a $w \in SO(3)$ with $w\Psi(x, s)w^{-1} = \Psi(y, u)$. Then $\Psi(wx, s) = \Psi(y, u)$ and so, by Lemma 3.11, (wx, s) and (y, u) lie on the same $N(\overline{T})$-orbit in $SO(3) \times N(\overline{T})^{2g}$. Therefore, the points $[(x, s)]$ and $[(y, u)]$ in $[SO(3) \times N(\overline{T})^{2g}]/N(\overline{T})$ lie on the same $SO(3)$ orbit, with $w \cdot [(x, s)] = [(y, u)]$. □

To understand the diagram (3.14b) at the smooth level we will show that the vertical arrow p corresponds to a smooth fiber bundle with fiber $SO(3)/\{I, \tau\}$, associated to a certain smooth principal bundle over $[N(\overline{T})^{2g}\backslash B]/N(\overline{T})$. The principal bundle will have the structure group $N(\overline{T})/\{I, \tau\}$. Having this, it clearly follows that the differentiable structure on $\mathrm{Im}\,\overline{\Psi}/SO(3)$ which makes $\overline{\overline{\Psi}}$ a diffeomorphism is the one which makes the quotient $p' : \mathrm{Im}\,\overline{\Psi} \to \mathrm{Im}\,\overline{\Psi}/SO(3)$ a submersion; consequently, with this differentiable structure, p' is a fiber-bundle projection.

The conjugation action of $N(\overline{T})$ on $N(\overline{T})^{2g}\backslash B$ has isotropy group $\{I, \tau\}$ everywhere, and so the quotient space $[N(\overline{T})^{2g}\backslash B]/N(\overline{T})$ is a smooth manifold and the projection $[N(\overline{T})^{2g}\backslash B] \to [N(\overline{T})^{2g}\backslash B]/N(\overline{T})$ is a principal $N(\overline{T})/\{I, \tau\}$-bundle.

Let

$$N'(\overline{T}) = N(\overline{T})/\{I, \tau\}. \tag{3.15a}$$

Note that $\{I, \tau\}$ is the center of $N(\overline{T})$.

Note also that $[N(\overline{T})^{2g}\backslash B]/N(\overline{T})$ is naturally diffeomorphic with $[N(\overline{T})^{2g}\backslash B]/N'(\overline{T})$, where the action of $N'(\overline{T})$ on $[N(\overline{T})^{2g}\backslash B]$ is simply the one induced by that of $N(\overline{T})$.

The smooth action of $N(\overline{T})$ on $SO(3)$ given by

$$(h, x) \mapsto xh^{-1} \tag{3.15b}$$

induces a smooth action of $N'(\overline{T})$ on $SO(3)/\{I, \tau\}$. Then we have the associated smooth fiber bundle

$$\begin{array}{l}\left(\dfrac{SO(3)}{\{I, \tau\}} \times (N(\overline{T})^{2g}\backslash B)\right)/N'(\overline{T}) \\ \downarrow \\ (N(\overline{T})^{2g}\backslash B)/N'(\overline{T}),\end{array}$$

where the quotient on top is with respect to the action of $N'(\overline{T})$ on $SO(3)/\{I,\tau\} \times (N(\overline{T})^{2g}\backslash B)$ given by

$$h \cdot (x\{I,\tau\}, p) = (xh^{-1}\{I,\tau\}, hph^{-1}). \tag{3.15c}$$

Note that this action is free and so the quotient is a smooth manifold.

The identity map

$$SO(3) \times (N(\overline{T})^{2g}\backslash B) \to SO(3) \times (N(\overline{T})^{2g}\backslash B)$$

induces a surjection

$$SO(3) \times (N(\overline{T})^{2g}\backslash B) \to \frac{SO(3)}{\{I,\tau\}} \times (N(\overline{T})^{2g}\backslash B),$$

which carries distinct $N(\overline{T})$-orbits onto distinct $N'(\overline{T})$-orbits. Thus there is a well-defined bijection

$$[SO(3) \times (N(\overline{T})^{2g}\backslash B)]/N(\overline{T}) \to \left[\frac{SO(3)}{\{I,\tau\}} \times (N(\overline{T})^{2g}\backslash B)\right]/N'(\overline{T}).$$

The two quotients here are with respect to free actions and so are smooth manifolds and the bijection above is a diffeomorphism.

We have the commutative diagram

$$\begin{array}{ccc} [SO(3) \times (N(\overline{T})^{2g}\backslash B)]/N(\overline{T}) & \to & \left[\dfrac{SO(3)}{\{I,\tau\}} \times (N(\overline{T})^{2g}\backslash B)\right]/N'(\overline{T}) \\ \downarrow p & & \downarrow p_1 \\ [N(\overline{T})^{2g}\backslash B]/N(\overline{T}) & \to & (N(\overline{T})^{2g}\backslash B)/N'(\overline{T}) \end{array} \tag{3.15d}$$

where the top and bottom arrows are diffeomorphisms and the vertical arrows are quotient maps. The important point here is that *the vertical arrow on the right is a fiber bundle;* it is the fiber bundle with fiber $SO(3)/\{I,\tau\}$ associated to the principal $N'(\overline{T})$-bundle $[N(\overline{T})^{2g}\backslash B] \to [N(\overline{T})^{2g}\backslash B]/N(\overline{T})$, where the structure group $N'(\overline{T})$ acts on the fiber $SO(3)/\{I,\tau\}$ in the manner induced by (3.15b).

Stringing together the two commutative diagrams (3.14b) and (3.15d), we obtain the commuting diagram:

$$\begin{array}{ccc} [SO(3) \times (N(\overline{T})^{2g}\backslash B)]/N'(\overline{T}) & \to & F \\ \downarrow p_1 & & \downarrow p' \\ [N(\overline{T})^{2g}\backslash B]/N'(\overline{T}) & \to & F/SO(3) \end{array} \tag{3.15e}$$

Here p_1 is a fiber bundle projection, p' is a quotient map, the top horizontal arrow is a diffeomorphism and the bottom horizontal arrow is a homeomorphism. Thus the differentiable structure on $F/SO(3)$ which makes the bottom arrow in (3.15e) (or, equivalently, in

(3.14b)) a diffeomorphism makes p' a submersion. We equip $F/SO(3)$ with this differentiable structure. Thus we have proved Proposition 3.10(b); in fact, we have:

Proposition 3.15. *Let F be the subset of $SO(3)^{2g}$ consisting of all points where the isotropy group of the $SO(3)$-action has two elements. Then the diagram*

$$\begin{array}{ccc} [SO(3)\times(N(\overline{T})^{2g}\backslash B)]/N(\overline{T}) & \stackrel{\overline{\Psi}}{\rightarrow} & F \\ \downarrow p & & \downarrow p' \\ [N(\overline{T})^{2g}\backslash B]/N(\overline{T}) & \stackrel{\overline{\overline{\Psi}}}{\rightarrow} & F/SO(3) \end{array} \tag{3.15f}$$

is an isomorphism, in the smooth category, of fiber bundles with fiber $SO(3)/\{I,\tau\}$ and structure group $N'(\overline{T}) \stackrel{\text{def}}{=} N(\overline{T})/\{I,\tau\}$, where τ is the $180°$ rotation belonging to the maximal torus $\overline{T}$. The bottom arrow is induced by the inclusion $N(\overline{T})^{2g}\backslash B \subset F$.

Furthermore, the fiber bundles given by p and p' are each isomorphic, in the smooth category, to the fiber bundle with fiber $SO(3)/\{I,\tau\}$ associated to the principal $N'(\overline{T})$-bundle given by the quotient $[N(\overline{T})^{2g}\backslash B] \to [N(\overline{T})^{2g}\backslash B]/N(\overline{T})$, where the action of the structure group $N'(\overline{T})$ on the fiber $SO(3)/\{I,\tau\}$ is the one induced by $h\cdot x = xh^{-1}$ for $h \in N(\overline{T})$, $x \in SO(3)$.

It will be useful to coordinatize $N(\overline{T})^{2g}$ as follows. Let J be a set of $2g$ elements, and view $\overline{T}^{2g}$ as $\overline{T}^{J}$. For $S \subset J$, we use the diffeomorphism

$$\phi_S : \overline{T}^{2g} \to N(\overline{T})^{2g} : (t_j)_{j\in J} \mapsto (\phi_S^j(t_j))_{j\in J}, \tag{3.16a}$$

where

$$\phi_S^j(x) = \begin{cases} x & \text{if} j \in S, \\ xn & \text{if} j \notin S. \end{cases} \tag{3.16b}$$

The sets $\phi_S(\overline{T}^{2g})$ are the different *components* of $N(\overline{T}^{2g})$.

We will use ϕ_S to transfer to $\overline{T}^{2g}$: (a) the conjugation action of $N(\overline{T})$ on $N(\overline{T})^{2g}$, and (b) the set B. Recall that B is the set of points in $\overline{T}^{2g}$ where the $SO(3)$-action has a two-element isotropy group.

Proposition 3.16.

(a) *Consider the action of $N(\overline{T})$ on $\overline{T}^{2g}$ given by (for $s \in \overline{T}$)*

$$s\cdot(t_j)_{j\in J} = (t_j')_{j\in J}, \quad \text{where} \quad t_j' = \begin{cases} t_j & \text{if } j \in S, \\ s^2 t_j & \text{if } j \notin S, \end{cases} \tag{3.16c}$$

and

$$sn\cdot(t_j)_{j\in J} = (t_j'')_{j\in J}, \quad \text{where} \quad t_j'' = \begin{cases} t_j^{-1} & \text{if } j \in S, \\ s^2 t_j^{-1} & \text{if } j \notin S. \end{cases} \tag{3.16d}$$

Then $\phi_S : \overline{T}^{2g} \to N(\overline{T})^{2g}$ is equivariant.

(b) *If* $S = J$ *then* $\phi_S(\overline{T}^{2g}) \subset B$; *if* $S \neq J$ *then* $\phi_S^{-1}(B)$ *is the orbit of the subset* $\{I, \tau\}^{2g}$ *under the action of* $N(\overline{T})$*:*

$$B_S \stackrel{\text{def}}{=} \phi_S^{-1}(B) = N(\overline{T}) \cdot \{I, \tau\}^{2g}.$$

(c) *If* S_1, S_2 *are distinct subsets of* J *then*

$$[Im(\phi_{S_1})/SO(3)] \cap [Im(\phi_{S_2})/SO(3)] \\ = [\phi_{S_1}(B_{S_1})/SO(3)] \cap [\phi_{S_2}(B_{S_2})/SO(3).$$

Proof.

(a) Readily verified by inspection.

(b) Recall from (3.11c) that $B = \overline{T}^{2g} \cup B'$, where B' is specified in (3.11d). If $S = J$, then ϕ_S is the inclusion map $\overline{T}^{2g} \to N(\overline{T})^{2g}$, and so $\phi_J(\overline{T}^{2g}) = \overline{T}^{2g} \subset B$.
Now suppose $S \neq J$. Consider a point $t = (t_j)_{j\in J} \in B_S$; let $\phi_S(t) = x = (x_j)_{j\in J}$. Then, since $S \neq J$, there is some $k \in J\backslash S$, and so $x_k = t_k n \in N(\overline{T})\backslash\overline{T}$, and so, in particular, $x \in B\backslash\overline{T}^{2g} = B'$. Therefore, by the definition of B' in (3.11d), $x_j \in \{I, \tau\}$ for every $j \in S$ and there is some $y \in \overline{T}$ such that $x_k \in \{yn, y\tau n\}$ for every $k \in J\backslash S$. Thus, $t_j \in \{I, \tau\}$ for every $j \in S$ and there is some $y \in \overline{T}$ such that $t_k \in \{y, y\tau\}$ for every $k \in J\backslash S$. Then t belongs to the $N(\overline{T})$-orbit through a point $t' \in \{I, \tau\}^{2g}$. Thus $B_S \subset N(\overline{T}) \cdot \{I, \tau\}^{2g}$.
Conversely, again with $S \neq J$, the isotropy group of the $N(\overline{T})$-action (as given in (3.16c) and (3.16d)) at any point of $\{I, \tau\}^{2g} \subset \overline{T}^{2g}$ is a four-element group (s or sn, where $s \in \overline{T}$, belongs to the isotropy group if and only if $s^2 = I$), and so no point on $N(\overline{T}) \cdot \{I, \tau\}^{2g}$ has isotropy group with exactly two elements, and so $N(\overline{T}) \cdot \{I, \tau\}^{2g} \subset B_S$.

(c) Suppose $\phi_{S_2}(t_j')_{j\in J} = x\phi_{S_1}(t_j)_{j\in J}x^{-1}$ for some $(t_j)_{j\in J}, (t_j')_{j\in J} \in \overline{T}^{2g}$, and $x \in SO(3)$. We shall show that $(t_j)_{j\in J} \in B_{S_1}$ and $(t_j')_{j\in J} \in B_{S_2}$. This will imply the desired result. In (b) we have seen that $(u_j) \in B_S$ means that $u_j \in \{I, \tau\}$ for all $j \in S$ and there is some $y \in \overline{T}$ such that $yu_k \in \{I, \tau\}$ for all $k \in J\backslash S$.

First we note that $x \notin N(\overline{T})$. For if x were an element of $N(\overline{T})$, then, picking $j \in S_1\backslash S_2$ (if this set is empty we can interchange S_1 with S_2, and t with t'), we would have $\phi_{S_{2j}}(t_j') = x\phi_{S_{1j}}(t_j)x^{-1} = t_j^{\pm 1} \in \overline{T}$, which is impossible since $\phi_{S_{2j}}(t_j') \in N(\overline{T})\backslash\overline{T}$ as $j \notin S_2$.

Let $j_* \in S_1 \cap S_2$; then $t_{j_*}' = xt_{j_*}x^{-1}$. Since $x \notin N(\overline{T})$, it follows from Observation 3.3(iii), that t_j and t_j' must be equal to I.

Consider $j \in S_1\backslash S_2$. Then $\phi_{S_{1j}}(t_j) = t_j \in \overline{T}$ while $\phi_{S_{2j}}(t_j') = t_j'n$ is a 180° rotation. So t_j, being conjugate to $t_j'n$, is the 180° rotation $\tau \in \overline{T}$. Similarly, $t_j' = \tau$ for all $j \in S_2\backslash S_1$.

Now consider $j, k \in J\backslash(S_1 \cup S_2)$. Writing out the conditions $x\phi_{S_{1j}}(t_j)x^{-1} = \phi_{S_{2j}}(t_j')$ and $x\phi_{S_{1k}}(t_k)x^{-1} = \phi_{S_{2k}}(t_k')$ we have $x(t_jn)x^{-1} = t_j'n$ and $x(t_kn)x^{-1} = t_k'n$. Then

$$x(t_jt_k^{-1})x^{-1} = t_j't_k'^{-1}.$$

Since $x \notin N(\overline{T})$, Observation 3.3(iii) implies that $t_j = t_k$. Thus there is a $y \in \overline{T}$ such that $t_j = y$ for all $j \in J\backslash(S_1 \cup S_2)$. Then $t'_j = \phi_{S_{2j}}(t'_j)n^{-1} = x\phi_{S_{1j}}(t_j)x^{-1}n^{-1} = xynx^{-1}n^{-1} = y'$, independent of the choice of j in $J\backslash(S_1 \cup S_2)$.

Consider $j \in S_2\backslash S_1$ and $k \in J\backslash(S_1 \cup S_2)$. Then

$$t'_j = \phi_{S_{2j}}(t'_j) = x\phi_{S_{1j}}(t_j)x^{-1} = xt_jnx^{-1}$$

and

$$t'_kn = \phi_{S_{2k}}(t'_k) = x\phi_{S_{1k}}(t_k)x^{-1} = xt_knx^{-1}.$$

So, using $(t'_kn)^{-1} = t'_kn$,

$$t'_jt'_kn = xt_jt_k^{-1}x^{-1}.$$

Now $t'_j = \tau$ since $j \in S_2\backslash S_1$, and $t'_k = y'$, independent of $k \in J\backslash(S_1 \cup S_2)$; so

$$t_jt_k^{-1} = x^{-1}(\tau y'n)x.$$

Thus $t_jt_k^{-1}$ is conjugate to a 180° rotation and therefore must be τ. Since $t_k = y$, independent of $k \in J\backslash(S_1 \cup S_2)$, we have $t_j = y\tau$ for every $j \in S_2\backslash S_1$.

Thus we have proved the following for $(t_j)_{j\in J}$: (i) if $j \in S_1$ then t_j is either I (if $j \in S_1 \cap S_2$) or τ (if $j \in S_1\backslash S_2$); (ii) there is a $y \in \overline{T}$ such that if $j \in J\backslash S_1$ then either $t_j = y$ (if $j \in J\backslash(S_1 \cup S_2)$) or $t_j = y\tau$ (if $j \in S_2\backslash S_1$). All of this simply says that $(t_j)_{j\in J} \in B_{S_1}$. Similarly, $(t'_j)_{j\in J} \in B_{S_2}$. □

3.5. *The structure of* $\overline{\mathcal{F}}_{2g-2}(\pm I)$

Recall (3.5a) that $\overline{\mathcal{F}}_{2g-2}(z) = \tilde{K}_g^{-1}(z) \cap F$, where F is the subset of $SO(3)^{2g}$ consisting of all points where the isotropy group of the $SO(3)$-action is a two-element group.

It will be convenient to take $N(\overline{T})^{2g}$ as $N(\overline{T})^J$, where J is the $2g$-element set

$$J = \{1, 2, 5, 6, \ldots, 4g-3, 4g-2\}.$$

With this notation,

$$\tilde{K}_g(p) = \prod_{j=1,5,\ldots,4g-3} \tilde{p}_{j+1}^{-1}\tilde{p}_j^{-1}\tilde{p}_{j+1}\tilde{p}_j, \tag{3.17a}$$

where $\tilde{p}_i$ is any element of $SU(2)$ which covers $p_i \in SO(3)$. (For $p \in N(\overline{T})$, each commutator appearing in the product above is actually an element of T.)

If $x, y \in N(\overline{T})$, then straightforward computation shows

$$\tilde{y}^{-1}\tilde{x}^{-1}\tilde{y}\tilde{x} = \begin{cases} I & \text{if } x, y \in \overline{T}, \\ \tilde{x}^2 & \text{if } x \in \overline{T} \text{ and } y \in N(\overline{T}), \\ \tilde{y}^{-2} & \text{if } x \in N(\overline{T}) \text{ and } y \in \overline{T}, \\ (\tilde{y}\tilde{x}^{-1})^2 & \text{if } x, y \in N(\overline{T}). \end{cases} \tag{3.17b}$$

Recall from (3.16a) and (3.16b) the charts ϕ_S parametrizing the components of $N(\overline{T})^{2g}$. We will use ϕ_S to transfer to $\overline{T}^{2g}$ the map $\tilde{K}_g$.

Proposition 3.17.

$$(\tilde{K}_g \circ \phi_S)(t_j)_{j\in J} = \prod_{j=1,5,\dots,4g-3} \tilde{t}_j^{m_j}\tilde{t}_{j+1}^{m_{j+1}} = \prod_{j\in J} \tilde{t}_j^{m_j}, \tag{3.17c}$$

where $\tilde{t}_j$ is any element of T covering $t_j \in \overline{T}$, and, for $j = 1, 5, \dots, 4g-3$,

$$(m_j, m_{j+1}) = \begin{cases} (0,0) & \text{if } j, j+1 \in S, \\ (2,0) & \text{if } j \in S \text{ and } j+1 \notin S, \\ (0,-2) & \text{if } j \notin S \text{ and } j+1 \in S, \\ (-2,2) & \text{if } j \notin S \text{ and } j+1 \notin S. \end{cases} \tag{3.17d}$$

Proof. Follows by combining (3.17a) and (3.17b). □

Recall that, for $z = \pm I$,

$$\overline{\mathcal{F}}_{2g-2}(z) = \tilde{K}_g^{-1}(z) \cap F \tag{3.18}$$

Proposition 3.18. *Suppose $g \geq 2$. Then $\overline{\mathcal{F}}_{2g-2}(\pm I)$ are $(2g+1)$-dimensional submanifolds of $SO(3)^{2g}$.*

The proof of this is contained in that of the next result, where we identify the components of $\overline{\mathcal{F}}_{2g-2}(\pm I)$:

Proposition 3.19. *Suppose $g \geq 2$. Then $\overline{\mathcal{F}}_{2g-2}(I)$ and $\overline{\mathcal{F}}_{2g-2}(-I)$ each have $2^{2g}-1$ connected components.*

Proof. Recall that $\overline{\mathcal{F}}_{2g-2}(z) = \tilde{K}_g^{-1}(z) \cap F$, where F is the set of points in $SO(3)^{2g}$ where the isotropy group of the $SO(3)$ action has two elements.

By Proposition 3.13, F is the diffeomorphic image under $\overline{\Psi}$ of the quotient $(SO(3) \times (N(\overline{T})^{2g}\backslash B))/N(\overline{T})$, the latter being a space with $2^{2g}-1$ components. Moreover, the space $\overline{\mathcal{F}}_{2g-2}(z) = \tilde{K}_g^{-1}(z) \cap F$ is diffeomorphic to the union of the $2^{2g}-1$ connected sets $(SO(3) \times ((\tilde{K}_g \circ \phi_S)^{-1}(z)\backslash B_S))/N(\overline{T})$, with S running over all proper subsets of the $2g$-element indexing set $J = \{1, 2, 5, 6, \dots, 4g-3, 4g-2\}$. Here $\phi_S : \overline{T}^{2g} \to N(\overline{T})^{2g}$ is the map given in (3.16a).

As we have noted,

$$(\tilde{K}_g \circ \phi_S)(t) = \prod_{j\in J} \tilde{t}_j^{m_j}, \tag{3.19a}$$

where $t = (t_j)_{j\in J} \in \overline{T}^{2g}$ is covered by $(\tilde{t}_j)_{j\in J} \in T^{2g}$, and $m_j \in \{0, \pm 2\}$ are as specified in (3.17d).

We work with a proper subset $S \subset J$. Fix $j_1 \in J$ such that $m_{j_1} \neq 0$ (by (3.17d) such j_1 exists). It is readily verified from (3.19a) that the restriction of the coordinate projection $\overline{T}^J \to \overline{T}^{J\setminus\{j_1\}}$ to $(\tilde{K}_g \circ \phi_S)^{-1}(z)$ is a bijection. Thus $(\tilde{K}_g \circ \phi_S)^{-1}(z)$ is diffeomorphic to $\overline{T}^{2g-1}$.

Since $\dim B_S = 1$ and $\dim(\tilde{K}_g \circ \phi_S)^{-1}(\pm I) = 2g-1$, and $g \geq 2$, it follows that each set $(\tilde{K}_g \circ \phi_S)^{-1}(z)\setminus B_S$ is connected and has dimension $2g-1$. The corresponding component of $\overline{\mathcal{F}}_{2g-2}(z)$ is

$$\overline{\mathcal{F}}_{2g-2}(z)_S = \text{union of all } SO(3)\text{-orbits through } \phi_S(\overline{T}^{2g}\setminus B_S) \cap \tilde{K}_g^{-1}(z). \tag{3.19b}$$

This is diffeomorphic to $(SO(3) \times ((\tilde{K}_g \circ \phi_S)^{-1}(z)\setminus B_S))/N(\overline{T})$, and therefore has dimension $2g+1$. □

3.6. *The quotient* $\overline{\mathcal{F}}_{2g-2}(\pm I) \to \overline{\mathcal{F}}_{2g-2}(\pm I)/SO(3)$

We have seen (in Proposition 3.15) that the quotient map $F \to F/SO(3)$ is a fiber bundle projection, where F is the subset of $SO(3)^{2g}$ consisting of all points where the isotropy group has two elements. For $z \in \{I, -I\}$, the set $\overline{\mathcal{F}}_{2g-2}(z)$ is, by Proposition 3.19, a submanifold of F, invariant under the action of $SO(3)$. Thus the bundle projection $F \to F/SO(3)$ restricts to a fiber bundle $\overline{\mathcal{F}}_{2g-2}(z) \to \overline{\mathcal{F}}_{2g-2}(z)/SO(3)$, with fiber $SO(3)/\{I, \tau\}$ (where τ is a 180° rotation) and structure group $N(\overline{T})/\{I, \tau\}$, where $\overline{T}$ is a maximal torus (containing τ) in $SO(3)$ and τ is the 180° rotation in $\overline{T}$. We set this out in detail in the following result.

Theorem 3.20. *Let z be I or $-I$. The quotient space $\overline{\mathcal{F}}_{2g-2}(z)/SO(3)$ is the union of $2^{2g}-1$ disjoint components. For any proper subset $S \subset J$, let $\overline{\mathcal{F}}_{2g-2}(z)_S$ be as in* (3.19b). *Then the sets $\overline{\mathcal{F}}_{2g-2}(z)_S/SO(3)$ are the $2^{2g}-1$ disjoint components of $\overline{\mathcal{F}}_{2g-2}(z)/SO(3)$. Moreover, for each proper subset S of J, there is a commutative diagram*

$$\begin{array}{ccc} \left[\dfrac{SO(3)}{\{I,\tau\}} \times \{(\tilde{K}_g \circ \phi_S)^{-1}(z)\setminus B_S\}\right]/N'(\overline{T}) & \overset{\psi_S}{\to} & \overline{\mathcal{F}}_{2g-2}(z)_S \\ \downarrow q & & \downarrow q' \\ [(\tilde{K}_g \circ \phi_S)^{-1}(z)\setminus B_S]/N'(\overline{T}) & \overset{\overline{\psi}_S}{\to} & \overline{\mathcal{F}}_{2g-2}(z)_S/SO(3) \simeq \mathcal{M}^0_{2g-2}(z)_S \end{array} \tag{3.20a}$$

in which the vertical arrows are quotient maps, and the horizontal arrows are diffeomorphisms. The vertical arrow given by q is the fiber bundle with fiber $SO(3)/\{I, \tau\}$ associated to the principal $N'(\overline{T})$-bundle given by the quotient map

$$[(\tilde{K}_g \circ \phi_S)^{-1}(z)\setminus B_S] \to [(\tilde{K}_g \circ \phi_S)^{-1}(z)\setminus B_S]/N'(\overline{T}), \tag{3.20b}$$

with $N'(\overline{T})$ acting on $SO(3)/\{I, \tau\}$ via conjugation, as in (3.15b)*. Thus the vertical arrow q' also specifies a fiber bundle with fiber $SO(3)/\{I, \tau\}$ and structure group $N'(\overline{T})$, and the diagram* (3.20a) *is an isomorphism of smooth fiber bundles in this category.*

The following gives an explicit description of the spaces $\overline{\mathcal{F}}_{2g-2}(z)_S/SO(3)$.

Proposition 3.21. *Let S be a proper subset of J. Let W be the two-element group $\{I, w\}$ acting on $\overline{T}^{2g-2}$ by $wx = x^{-1}$. There is a smooth one-to-one map*

$$j_S : \overline{T}^{2g-2} \to SO(3)^{2g}$$

such that
(i) $\det dj_S$ *is constant ($\neq 0$) everywhere on* $\overline{T}^{2g-2}$,
(ii) $j_S(\overline{T}^{2g-2}\backslash\{I,\tau\}^{2g}) \subset \overline{\mathcal{F}}_{2g-2}(z)_S$,
(iii) j_S *induces a diffeomorphism* $\overline{j}_S : (\overline{T}^{2g-2}\backslash\{I,\tau\}^{2g})/W \to \overline{\mathcal{F}}_{2g-2}(z)_S/SO(3)$.

Proof. Since S is a proper subset of J, the specification of the m_j given in (3.17d) allows us to choose distinct $j_1, j_2 \in J$ such that $m_{j_1} \neq 0$ and $j_2 \notin S$. Let

$$j'_S : \overline{T}^{J\backslash\{j_1,j_2\}} \to \overline{T}^{2g} : x \mapsto x'$$

be specified by

$$x'_j = \begin{cases} x_j & \text{if } j \in J\backslash\{j_1, j_2\}, \\ I & \text{if } j = j_2, \\ \prod_{j\in J\backslash\{j_1,j_2\}} x_j^{-m_j/m_{j_1}} & \text{if } j = j_1 \text{ and } z = I, \\ \tau \prod_{j\in J\backslash\{j_1,j_2\}} x_j^{-m_j/m_{j_1}} & \text{if } j = j_1 \text{ and } z = -I, \end{cases},$$

where τ is the 180° rotation belonging to $\overline{T}$. Note that $m_j/m_{j_1} \in \{0, \pm 1\}$. Then we define

$$j_S = \phi_S \circ j'_S.$$

The definition of j'_S shows that $\mathrm{d}j'_S(X) = X' = (X'_j)_{j\in J}$, where

$$X'_j = \begin{cases} X_j & \text{if } j \in J\backslash\{j_1, j_2\}, \\ 0 & \text{if } j = j_2, \\ -\sum_{j\in J\backslash\{j_1\}} \frac{m_j}{m_{j_1}} X_j & \text{if } j = j_1. \end{cases}$$

It follows from this (or from the corresponding expression for $\mathrm{d}j_S'^{*}\mathrm{d}j'_S$) that

$$\det \mathrm{d}j'_S = \sqrt{1 + \sum_{j\in J\backslash\{j_1,j_2\}} \frac{m_j^2}{m_{j_1}^2}}$$

(the specification of the m_j given in (3.17d) shows that $\det \mathrm{d}j'_S = \sqrt{2g - \#S - |m_{j_2}|/2}$. Since ϕ_S is an isometry, $\det \mathrm{d}j_S = \det \mathrm{d}j'_S$.

By (3.19a), we have $(\tilde{K}_g \circ \phi_S)(x) = \prod_{j\in J} \tilde{x}_j^{m_j}$, where $\tilde{x}_j \in T$ covers $x_j \in \overline{T}$. Using the definition of the x'_j, and the fact that $\tilde{\tau}^2 = -I$, we see then that

$$\tilde{K}_g \circ j_S(x) = (\tilde{K}_g \circ \phi_S)(j'_S(x)) = (\tilde{K}_g \circ \phi_S)(x') = z.$$

Since $j_2 \notin S$ and the j_2th component of any element in the image of j'_S is, by definition, I, it follows, that for any $x \in \overline{T}^{2g-2}$, the image $j'_S(x)$ lies in B_S if and only if $x \in \{I, \tau\}^{2g}$. Thus j_S maps $\overline{T}^{2g-2}\backslash\{I, \tau\}^{2g}$ into $\overline{\mathcal{F}}_{2g-2}(z)_S$.

If two points in $j_S(\overline{T}^{2g-2}\backslash\{I, \tau\}^{2g})$ are on the same $SO(3)$-orbit then the corresponding points in $j'_S(\overline{T}^{2g-2}\backslash\{I, \tau\}^{2g})$ are on the same $N(\overline{T})$-orbit (this follows from Lemma 3.11). Examination of Proposition 3.16(a) then shows that ($s^2 = 1$ in (3.16c)) the points in $\overline{T}^{2g-2}\backslash\{I, \tau\}^{2g}$ are on the same W-orbit. Thus j_S quotients to a one-to-one map

$$\overline{j_S} : (\overline{T}^{2g-2}\backslash\{I, \tau\}^{2g})/W \to \overline{\mathcal{F}}_{2g-2}(z)_S/SO(3).$$

If $y \in \overline{\mathcal{F}}_{2g-2}(z)_S$ then by appropriate conjugation we can assume that $y \in \phi_S(\overline{T}^{2g})$ and $y_{j_2} = n$. Then the point $x' = \phi_S^{-1}(y)$ has $x_{j_2} = I$. Since $\tilde{K}_g \circ \phi_S(x') = z$, the component x'_{j_1} is determined by the other components, and it follows that x' lies in the image of j_S. Thus $\overline{j_S}$ is also surjective.

Since j_S is an immersion, so is $\overline{j_S}$. Moreover, $\overline{j_S}$ is a homeomorphism of $(\overline{T}^{2g-2}\backslash\{I, \tau\}^{2g})/W$ onto its image (the fact that $\overline{j_S}|(\overline{T}^{2g-2}\backslash\{I, \tau\}^{2g})/W$ is a closed map can be verified using the observation we made above that a point $x \in \overline{T}^{2g-2}$ in the image of j_S lies in $\overline{\mathcal{F}}_{2g-2}(z)_S$ if and only if $x \in \overline{T}^{2g-2}\backslash\{I, \tau\}^{2g}$). Combining all these, we see that $\overline{j_S}$ is a diffeomorphism of $(\overline{T}^{2g-2}\backslash\{I, \tau\}^{2g})/W$ onto its image. □

3.7. *The sets $\overline{\mathcal{F}}_0(z)$ and $\overline{\mathcal{F}}_0(z)/SO(3)$*

Recall (from (3.5a)) that $\overline{\mathcal{F}}_0(z)$ is the subset of $\tilde{K}_g^{-1}(z)$ where the isotropy group is either $SO(3)$ or $N(\overline{T})$, the normalizer of a maximal torus $\overline{T}$ in $SO(3)$, or is of the form $\{I, \tau_1, \tau_2, \tau_3\}$ for some 180° rotations τ_1, τ_2, τ_3 around orthogonal axes.

Let

$$F_0 = \begin{cases} \text{the subset of}\, SO(3)^{2g}\, \text{consisting of all points where the} \\ \text{isotropy group is either}\, SO(3) \\ \text{or the normalizer of a maximal torus in}\, SO(3), \\ \text{or a four-element group.} \end{cases} \tag{3.21a}$$

These cases are covered by Proposition 3.4(i)–(iii), from where we see that a point $(x_1, \ldots, x_{2g}) \in SO(3)^{2g}$ belongs to F_0 if and only if $\{x_1, \ldots, x_{2g}\} \subset \{I, n_1, n_2, n_3\}$, where n_1, n_2, n_3 are 180° rotations around three orthogonal axes. Thus, fixing 180° rotations τ_1, τ_2, τ_3 around three orthogonal axes, we have

$$F_0 = \bigcup_{x \in SO(3)} xF'_0x^{-1}, \quad \text{where } F'_0 = \{I, \tau_1, \tau_2, \tau_3\}^{2g}. \tag{3.21b}$$

Let S_3 be the group of permutations on $\{I, \tau_1, \tau_2, \tau_3\}$ which fix I; thus S_3 has a natural action on F'_0. Two points in F'_0 lie in the same S_3-orbit if and only if they lie in the same $SO(3)$-orbit in F_0 (every permutation of $\{\tau_1, \tau_2, \tau_3\}$ can be realized as the conjugation by some element of $SO(3)$, since the permutation $\tau_1 \leftrightarrow \tau_2$ is realized by conjugation by $\tau_3^{1/2}$ – a 90° rotation around the axis for τ_3). Thus we have a bijection

$$F_0/SO(3) \simeq F_0'/S_3 \tag{3.21c}$$

induced by the inclusion $F_0' \subset F_0$.

Proposition 3.22. *The sets F_0 and F_0' split into the following disjoint sets according to isotropy type:*

$$F_0 = F_{00} \cup F_{01} \cup F_{02} \quad \text{and} \quad F_0' = F_{00}' \cup F_{01}' \cup F_{02}', \tag{3.21d}$$

where $F_{0j}' = F_{0j} \cap \{I, \tau_1, \tau_2, \tau_3\}^{2g}$, and

(i) *$F_{00} = F_{00}'$ is the singleton consisting of the point $(I, I, \ldots, I)$, and the isotropy groups are the full groups.*

(ii) *F_{01} is the set of points where the isotropy group is the normalizer of a maximal torus in $SO(3)$, and $F_{01}' = \bigcup_{j=1}^3 \{I, \tau_j\}^{2g} \backslash \{(I, I, \ldots, I)\}$ is the set of points in F_0' where the isotropy group is a two-element subgroup of S_3. Each $SO(3)$ orbit through a point of the set F_{01} is equivariantly diffeomorphic to the connected 2-dimensional space $SO(3)/N(K)$, where $N(K)$ is the normalizer of the maximal torus K in $SO(3)$. The number of components of F_{01} is*

$$\#F_{01}/SO(3) = \#F_{01}'/S_3 = 2^{2g} - 1. \tag{3.21e}$$

(iii) *F_{02} is the set of points where the isotropy group is a four-element group, and $F_{02}' = F_0' \backslash \bigcup_{j=1}^3 \{I, \tau_j\}^{2g}$ is the subset of F_0' where the isotropy group is trivial. Each orbit through F_{02} is equivariantly diffeomorphic to the connected 3-manifold $SO(3)/\{I, \tau_1, \tau_2, \tau_3\}$. The number of connected components of F_{02} is*

$$\#F_{02}/SO(3) = \#F_{02}'/S_3 = \#F_{02}' = \tfrac{1}{6}(4^{2g} - 3 \cdot 2^{2g} + 2). \tag{3.21f}$$

The total number of components of F_0 is

$$\#F_0/SO(3) = \#F_0'/S_3 = \#\tfrac{1}{6}(4^{2g} + 3 \cdot 2^{2g} + 2). \tag{3.21g}$$

Proof. The decomposition of F_0 according to isotropy is provided by Proposition 3.4(i)–(iii), which also shows that F_{0j} consists of the points in the orbits through F_{0j}'. Inspection shows that the isotropy group (in S_3) at each point of F_{01}' is the two-element group generated by a transposition $\tau_i \leftrightarrow \tau_j$, while the isotropy group in S_3 at each point of F_{02}' is trivial. Since $\#F_{01}' = 3(2^{2g} - 1)$, and the isotropy at each point has two elements, we obtain (3.21e). Next,

$$\#F_{02}' = \#F_0' - \#F_{00}' - \#F_{01}' = 4^{2g} - 1 - 3(2^{2g} - 1) = 4^{2g} - 3 \cdot 2^g + 2,$$

and so, since S_3 acts freely on $\#F_{02}'$, we have $\#F_{02}'/S_3$ is $\frac{1}{6}$th of $\#F_{02}'$. Finally, $\#F_0/S_3$ is the sum of the $\#F_{0j}'/S_3$. □

We are interested in the set

$$\overline{\mathcal{F}}_0(z) = F_0 \cap \tilde{K}_g^{-1}(z), \tag{3.22a}$$

and the quotient

$$\mathcal{M}_0^0(z) = \overline{\mathcal{F}}_0(z)/SO(3) \simeq F_0' \cap \tilde{K}_g^{-1}(z)/S_3. \tag{3.22b}$$

The set $\overline{\mathcal{F}}_0(z)$ is the union of the subsets $F_{0j} \cap \tilde{K}_g^{-1}(z)$.

For the purpose of counting, we shall view a point of $\{I, \tau_1, \tau_2, \tau_3\}^{2g}$ as a g-tuple of pairs $(a_i, b_i) \in \{I, \tau_1, \tau_2, \tau_3\}^2$.

By Observation 3.3(ii), for $(a, b) \in \{I, \tau_1, \tau_2, \tau_3\}^2$ (with $\tilde{x}$ denoting, as usual, any element of $SU(2)$ covering $x \in SO(3)$)

$$\tilde{a}\tilde{b}\tilde{a}^{-1}\tilde{b}^{-1} = \begin{cases} -I & \text{if } a \text{ and } b \text{ are distinct elements of } \{\tau_1, \tau_2, \tau_3\}, \\ I & \text{otherwise.} \end{cases}$$

Let us say that a pair $(a, b) \in \{I, \tau_1, \tau_2, \tau_3\}^2$ is *positive* if $\tilde{a}\tilde{b}\tilde{a}^{-1}\tilde{b}^{-1} = I$, and *negative* if $\tilde{a}\tilde{b}\tilde{a}^{-1}\tilde{b}^{-1} = -I$. Of the 16 elements in $\{I, \tau_1, \tau_2, \tau_3\}^2$, 6 are negative and 10 are positive.

It is readily seen that for a point $p = (p_1, \dots, p_g) \in F_0'$,

$$\begin{aligned} &p \in F_0' \cap \tilde{K}_g^{-1}(I) && \text{if } \#\{j : p_j \text{ is negative}\} \text{ is even,} \\ &p \in F_0' \cap \tilde{K}_g^{-1}(-I) && \text{if } \#\{j : p_j \text{ is negative}\} \text{ is odd.} \end{aligned}$$

Thus the total number of points in $F_0' \cap \tilde{K}_g^{-1}(I)$ is the sum of the coefficients of the even powers of x in the polynomial $(10 + 6x)^g$, while $\#F_0' \cap \tilde{K}_g^{-1}(-I)$ is the sum of the coefficients of the odd powers of x in the polynomial $(10 + 6x)^g$:

$$\#F_0' \cap \tilde{K}_g^{-1}(I) = \tfrac{1}{2}(16^g + 4^g), \qquad \#F_0' \cap \tilde{K}_g^{-1}(-I) = \tfrac{1}{2}\left(16^g - 4^g\right). \tag{3.22c}$$

It is clear that $F_{00}' \cup F_{01}' \subset \tilde{K}_g^{-1}(I)$. So

$$\begin{aligned} \#F_{02}' \cap \tilde{K}_g^{-1}(I) &= \#F_0' \cap \tilde{K}_g^{-1}(I) - \#F_{00}' - \#F_{01}' \\ &= \tfrac{1}{2}\left(16^g + 4^g\right) - 1 - 3(2^{2g} - 1), \end{aligned} \tag{3.22d}$$

and

$$F_0' \cap \tilde{K}_g^{-1}(-I) = F_{02}' \cap \tilde{K}_g^{-1}(-I). \tag{3.22e}$$

Combining all these observations, we obtain:

Theorem 3.23.

(i) *$\overline{\mathcal{F}}_0(I)$ is the union of disjoint $SO(3)$-invariant subsets*

$$\overline{\mathcal{F}}_0(I) = \overline{\mathcal{F}}_{00}(I) \cup \overline{\mathcal{F}}_{01}(I) \cup \overline{\mathcal{F}}_{02}(I),$$

where $\overline{\mathcal{F}}_{00}(I) = \{(I, I, \dots, I)\}$, $\overline{\mathcal{F}}_{01}(I)$ is the subset consisting of points where the isotropy group is the normalizer of a maximal torus in $SO(3)$, and $\overline{\mathcal{F}}_{02}(I)$ is the subset consisting of points where the isotropy is a four-element group.

(ii) $\overline{\mathcal{F}}_{01}(I)$ *is a two-dimensional submanifold of* $SO(3)^{2g}$. *The quotient* $\overline{\mathcal{F}}_{01}(I)/SO(3)$ *is a finite set, and each fiber of the projection* $\overline{\mathcal{F}}_{01}(I) \to \overline{\mathcal{F}}_{01}(I)/SO(3)$ *is diffeomorphic to* $SO(3)/N(K)$, *where* $N(K)$ *is the normalizer of any maximal torus* K *in* $SO(3)$.

(iii) $\overline{\mathcal{F}}_{02}(I)$ *is a three-dimensional submanifold of* $SO(3)^{2g}$. *The quotient* $\overline{\mathcal{F}}_{02}(I)/SO(3)$ *is a finite set, and each fiber of the projection* $\overline{\mathcal{F}}_{02}(I) \to \overline{\mathcal{F}}_{02}(I)/SO(3)$ *is diffeomorphic to* $SO(3)/\{I, \tau_1, \tau_2, \tau_3\}$, *where* τ_1, τ_2, τ_3 *are* 180° *rotations around orthogonal axes.*

(iv) $\overline{\mathcal{F}}_0(-I)$ *is a three-dimensional submanifold of* $SO(3)^{2g}$. *The quotient* $\overline{\mathcal{F}}_0(-I)/SO(3)$ *is a finite set, and each fiber of the projection* $\overline{\mathcal{F}}_0(-I) \to \overline{\mathcal{F}}_0(-I)/SO(3)$ *is diffeomorphic to* $SO(3)/\{I, \tau_1, \tau_2, \tau_3\}$, *where* τ_1, τ_2, τ_3 *are* 180° *rotations around orthogonal axes.*

Focusing on the quotients $\overline{\mathcal{F}}_0(z)/SO(3)$, we have:

Theorem 3.24. *The sets* $\mathcal{M}_0^0(I)$ *and* $\mathcal{M}_0^0(-I)$ *are discrete, and*

$$\#\mathcal{M}_0^0(I) = \tfrac{1}{12}[2^{4g} + 7 \cdot 2^{2g} + 4], \quad \#\mathcal{M}_0^0(-I) = \tfrac{1}{12}[16^g - 4^g].$$

Proof. $\#\mathcal{M}_0^0(I) = \#\overline{\mathcal{F}}_0(I)/SO(3) = \#F_0' \cap \tilde{K}_g^{-1}(I)/SO(3)$ is obtained by adding up the $\#F_{0j}' \cap \tilde{K}_g^{-1}(I)/SO(3)$ (which are given in (3.22c) and (3.22d). For $\mathcal{M}_0^0(-I) = \overline{\mathcal{F}}_0(-I)/SO(3) = F_0' \cap \tilde{K}_g^{-1}(-I)/SO(3)$, we use (3.22e) and (3.22c). □

4. Some technical facts

In this section we record some technical facts used elsewhere in this paper.

Lemma 4.1. *Let* X, Y *be vector spaces, and* $L_1, L_2 : X \to Y$ *surjective linear maps such that*

$$\ker(L_1) + \ker(L_2) = X. \tag{4.1a}$$

Then

$$L_1([\ker(L_1 + L_2)]) = Y. \tag{4.1b}$$

Proof. Condition (4.1a), together with the fact that L_1 and L_2 are surjective, implies that L_1 maps $\ker L_2$ onto Y. Similarly, $L_2(\ker L_1) = y$. Let $y \in Y$. We can choose $x_1 \in \ker L_2$ and $x_2 \in \ker L_1$ such that $L_1 x_1 = y$ and $L_2 x_2 = -Y$. Let $x = x_1 + x_2$. Then $L_1 x = y$ and $L_2 x = -y$. So $x \in \ker(L_1 + L_2)$. □

Application of Lemma 4.1. We used Lemma 4.1 in the proofs of Proposition 2.7. Let $g \geq 2$, and consider the maps $C_r : G^{2g} \to G : (x_1, y_1, \ldots, x_g, y_g) \mapsto y_r^{-1} x_r^{-1} y_r x_r$, and $K = C_g \ldots C_1$, and $K' = C_g \ldots C_2$. We will show that C_1 restricted to the submanifold

$\mathcal{F}^1(h) = C_1^{-1}(G\backslash\{I,h\}) \cap K_g^{-1}(h)$ is a submersion, for any $h \in G$. Working at a fixed point on $\mathcal{F}^1(h)$, let

$$L_1 = C_1^{-1}\mathrm{d}C_1, \quad L_2 = (\mathrm{Ad}\,C_1^{-1})K'^{-1}\mathrm{d}K'.$$

Then $\ker L_2 \supset \underline{g} \oplus \underline{g} \oplus \{0\} \oplus \cdots \oplus \{0\}$, and $\ker L_1 \supset \{0\} \oplus \{0\} \oplus \underline{g} \oplus \cdots \oplus \underline{g}$, and so $\ker L_1 + \ker L_2 = \underline{g}^{2g}$. Moreover, by Lemma 2.4(ii), at any point in $\mathcal{F}^1(h)$, L_1 and L_2 are surjective. Using $K = K'C_1$, we have $K^{-1}\mathrm{d}K = L_1 + L_2$. So, by Lemma 4.1, this implies that $L_1 | \ker(K^{-1}\mathrm{d}K)$ is surjective. Since $\ker(K^{-1}\mathrm{d}K)$ is the (left-translated) tangent space to $\mathcal{F}^1(h)$, we conclude that $C_1|\mathcal{F}^1(h)$ is a submersion.

4.1. Group actions on manifolds

We have used the following result several times:

Proposition 4.2. *Let G be a compact Lie group, M a smooth manifold, $M \times G \to M : (m,g) \mapsto mg$ a free smooth right action, and let $p : M \to M/G$ be the corresponding quotient map onto the quotient space M/G. Then there is a (unique) smooth manifold structure on M/G for which p is a submersion; with this structure on M/G, the projection $p : M \to M/G$, along with the action of G on M, is a smooth principal G-bundle.*

This result is proved in [1, 16.14.1 and 16.10.3] ([1, 16.10.3] is stated with the hypothesis that $\{(m, mg) : m \in M, g \in G\}$ is a closed submanifold of $M \times M$; this condition may be verified by examining the map $f : M \times G \to M \times M : (m,g) \mapsto (m, mg)$ and using the compactness of G along with the hypothesis that the action of G on M is free; f is a smooth one-to-one immersion and its image is closed in M^2).

Lemma 4.3. *Let G be a compact Lie group acting smoothly and isometrically on a Riemannian manifold M:*

$$G \times M \to M : (x,m) \mapsto \gamma_m(x) = xm.$$

Suppose that the isotropy group is the same subgroup $H \subset G$ at every point of M. Fix an Ad-invariant metric on the Lie algebra $\underline{g}$ of G, and let $\underline{h}$ be the Lie algebra of H. Let $\mathrm{d}\gamma_m : \underline{g} \to T_mM$ be the derivative of γ_m at the identity in G. Then

$$m \mapsto |\det(\mathrm{d}\gamma_m|\underline{h}^\perp : \underline{h}^\perp \to T_mM)| \tag{4.2a}$$

is a G-invariant function of m, thus defining a function $|\det d\gamma|\underline{h}^\perp|$ on M/G.

If f is any G-invariant Borel function on M, then

$$\int_M f\,\mathrm{dvol}_M = \mathrm{vol}\,(G/H) \int_{M/G} \tilde{f}\,|\det \mathrm{d}\gamma|\underline{h}^\perp|\,\mathrm{dvol}_{M/G} \tag{4.2b}$$

(either side existing if the other does) where vol *denotes Riemannian volume on the appropriate spaces (taken as counting measure when the space is discrete), and $\tilde{f}$ is the function*

on M/G induced by f. (In particular, if H is finite then (4.3b) *holds with* vol $(G)/\#H$ *for* vol (G/H)).

Proof. We shall denote the action of the derivative of $m \mapsto xm$ on $v \in T_m M$ by $x \cdot v$. From $\gamma_{ym}(x) = y\gamma_m(y^{-1}xy)$, we have $\mathrm{d}\gamma_{ym} = y \cdot \mathrm{d}\gamma_m \circ \mathrm{Ad}(y^{-1})$; thus (4.2a) is G-invariant since the G action $m \mapsto ym$ is an isometry and since the metric on $\underline{g}$ is Ad-invariant.

The isotropy group H being the same everywhere, it follows that H is a normal (closed) subgroup of G. The induced action of the group G/H on M is smooth and free, and therefore, by Proposition 4.2, $M/G \simeq M/(G/H)$ is a smooth manifold and the quotient map $\pi : M \to M/G$ specifies a smooth principal G/H-bundle. Consider then a G-equivariant diffeomorphism

$$(G/H) \times U \xrightarrow{\psi} \pi^{-1}(U), \tag{4.3a}$$

where U is a non-empty open subset of M/G, and $\pi\psi(xH, u) = u$ for every $u \in U$ and $x \in G$. Note that G-equivariance means that $\psi(gxH, u) = \gamma_m(g)$ where $m = \psi(xH, u)$. We split the tangent space $T_m M$ into orthogonal subspaces (note that $\underline{h}^{\perp}$ corresponds to the Lie algebra of G/H) :

$$T_m M = \mathrm{d}\gamma_m(\underline{h}^{\perp}) + \mathrm{d}\gamma_m(\underline{h}^{\perp})^{\perp} \simeq \mathrm{d}\gamma_m(\underline{h}^{\perp}) \oplus T_u(M/G), \tag{4.3b}$$

where the $\simeq$ is obtained from the unitary isomorphism $[\mathrm{d}\gamma_m(\underline{h}^{\perp})]^{\perp} \to T_u(M/G)$ given by $\mathrm{d}\pi$ (the condition that this restriction of $\mathrm{d}\pi$ is unitary defines the metric on M/G). Thus the matrix of $\mathrm{d}\psi_{(xH,u)}$ has the form

$$\begin{bmatrix} \mathrm{d}\gamma_m|\underline{h}^{\perp} & * \\ 0 & I \end{bmatrix}. \tag{4.3c}$$

Consequently,

$$|\det \mathrm{d}\psi|_{(xH,u)}| = |\det(\mathrm{d}\gamma_m|\underline{h}^{\perp})|. \tag{4.3d}$$

It follows that Eq. (4.3b) holds for f supported in $\pi^{-1}(U)$. By using a partition of unity argument it follows that (4.3b) holds for all compactly supported continuous G-invariant functions f. Then, by definition of the measures vol_M and $\mathrm{vol}_{M/G}$, Eq. (4.3b) holds for any G-invariant Borel function $f \geq 0$, and hence for any Borel f for which either side of (4.3b) exists. □

5. The symplectic structure

We work with a principal G-bundle $\pi : P \to \Sigma$ over a closed oriented surface Σ of genus $g \geq 1$, where the structure group G is $SU(2)$ or $SO(3)$, equipped with an Ad-invariant metric. There is a standard symplectic structure Ω on the infinite-dimensional space $\mathcal{A}$ of connections on P. The action on $\mathcal{A}$ of the group $\mathcal{G}$ of bundle automorphisms preserves the symplectic structure, and there is a moment map J whose value $J(\omega)$, for any $\omega \in \mathcal{A}$, can be identified with the curvature of ω. The Marsden–Weinstein procedure then yields, formally, a 2-form $\overline{\Omega}$ on the moduli space of flat connections $\mathcal{M}^0 = J^{-1}(0)/\mathcal{G}$ (a rigorous account of this presented in [7]). Now let $A_1, B_1, \ldots, A_g, B_g$ be standard loops generated $\pi_1(\Sigma, o)$, where o is a fixed basepoint on Σ and $\overline{B}_g\overline{A}_g B_g A_g \cdots \overline{B}_g\overline{A}_g B_g A_g$ is the identity in $\pi_1(\Sigma, o)$. Denoting by $h(C;\omega)$ the holonomy of a connection ω around a loop C based at o (using a fixed reference point on the fiber $\pi^{-1}(o)$), we have the map

$$\mathcal{H} : \mathcal{A} \to G^{2g} : \omega \mapsto (h(A_1;\omega), h(B_1;\omega), \ldots, h(A_g;\omega), h(B_g;\omega)).$$

This map carries the set $\mathcal{A}^0$ of flat connections onto the subset $\tilde{K}_g^{-1}(z)$, where

$$\tilde{K}_g : G^{2g} \to \tilde{G} : (a_1, b_1, \ldots, a_g, b_g) \mapsto \tilde{b}_g^{-1}\tilde{a}_g^{-1}\tilde{b}_g\tilde{a}_g \cdots \tilde{b}_1^{-1}\tilde{a}_1^{-1}\tilde{b}_1\tilde{a}_1,$$

with $\tilde{x}$ denoting any element in the universal cover $\tilde{G}$ of G projecting to $x \in G$, and z is a certain element of $\ker(\tilde{G} \to G)$ which characterizes the topology of the bundle P. In fact, $\mathcal{H}$ induces a bijection

$$\overline{\mathcal{H}} : \mathcal{A}^0/\mathcal{G} \to \tilde{K}_g^{-1}(z)/G,$$

where the quotient on the right is with respect to the action of G given by conjugation of each coordinate in G^{2g}. We will always identify $\mathcal{M}^0 = \mathcal{A}^0/\mathcal{G}$ with $\tilde{K}_g^{-1}(z)/G$ in this way. There is a 2-form Ω' on G^{2g} whose restriction to $\tilde{K}_g^{-1}(z)$ is the lift of the 2-form $\overline{\Omega}$ mentioned earlier.

We will work with the group G^{2g}, where $g \geq 1$ and G is either $SU(2)$ or $SO(3)$. It will be useful to label the coordinates of a point of G^{2g} with subscripts in the following way; let

$$J = \{1, 2, 5, 6, \ldots, 4g-3, 4g-2\}. \tag{5.1a}$$

Thus J is a set with $2g$ elements; we shall take a typical point of G^{2g} to be $(\alpha_j)_{j\in J}$. We then define α_i, for $i \in \{3, 4, 7, 8, \ldots, 4g-1, 4g-2\} = J+2$ by

$$\alpha_{j+2} = \alpha_j^{-1} \quad \text{for all } j \in J. \tag{5.1b}$$

A vector in the tangent space $T_\alpha G^{2g}$ then has the form $\alpha \cdot H$, where $H \in \underline{g}^{2g}$ has components $(H_j)_{j\in J}$; we set

$$H_{j+2} = -\mathrm{Ad}(\alpha_j)H_j \quad \text{for all } j \in J. \tag{5.1c}$$

The 2-form Ω' on G^{2g}, defined on vectors $\alpha W, \alpha Z \in T_\alpha G^{2g}$ by

$$\Omega'(\alpha W, \alpha Z) = \frac{1}{2} \sum_{1\leq i,k\leq 4g} \epsilon_{ik} \langle f_{i-1}^{-1} W_i, f_{k-1}^{-1} Z_k \rangle, \tag{5.2a}$$

where $f_i = \mathrm{Ad}\,(\alpha_i \ldots \alpha_1)$ for each $i \in \{1, \ldots, 4g\}$, f_0 is the identity map, and

$$\epsilon_{ik} = \begin{cases} 1 & \text{if } i < k, \\ -1 & \text{if } i > k, \\ 0 & \text{if } i = k. \end{cases} \tag{5.2b}$$

By appropriate left-translation, the derivative of K_g at α may be taken to be a map $\mathrm{d}K_g : \underline{g}^{2g} \to \underline{g}$; denote by $\mathrm{d}K_g(\alpha)^* : \underline{g} \to \underline{g}^{2g}$ its adjoint with respect to the metric on $\underline{g}$.

Here are some useful properties of Ω' (proofs may be found in [4] or [7]):

Proposition 5.1.

(i) *Ω' is G-invariant.*

(ii) *$\Omega'_p(A, B)$ is 0 if $A \in T_pG^{2g}$ is tangent to a smooth path lying on $\tilde{K}_g^{-1}(z)$ and B is tangent to the G-orbit through p.*

(iii) *$d\Omega'(A, B) = 0$ if A, B are tangent to $\tilde{K}_g^{-1}(z)$.*

(iv) *Let $\gamma_\alpha : G \to \tilde{K}_g^{-1}(z) : x \mapsto x\alpha x^{-1}$ be the orbit map. Recall the product commutator map $\tilde{K}_g : G^{2g} \to \tilde{G}$. If $\alpha \in \tilde{K}_g^{-1}(z)$ then*

$$\Omega'_b \circ \mathrm{d}\gamma_\alpha = \mathrm{d}\tilde{K}_g(\alpha)^*, \tag{5.3}$$

where Ω'_b is specified by $\Omega'(X, Y) = \langle X, \Omega'_b Y\rangle$.

Eq. (5.3) says that $\mathrm{d}\tilde{K}_g^*$ is like a moment map.

Recall that when $G = SU(2)$, $K_g^{-1}(I)$ is the union of manifolds $\mathcal{F}_{3(2g-2)}, \mathcal{F}_{2g}, \mathcal{F}_0$, while for $G = SO(3)$, $\tilde{K}_g^{-1}(z)$ is the union of manifolds $\overline{\mathcal{F}}_{3(2g-2)}(z), \overline{\mathcal{F}}_{2g}(z), \overline{\mathcal{F}}_{2g-2}(z), \overline{\mathcal{F}}_0(z)$, where $\overline{\mathcal{F}}_{2g}(z)$ is empty if $z = -I$. The corresponding quotients under the conjugation action of G are denoted $\mathcal{M}_k^0(z)$ (if $G = SU(2)$, z can only be I and we drop it from the notation sometimes), with $k \in \{3(2g-2), 2g, 2g-2, 0\}$.

Proposition 5.2. *There is a unique smooth closed 2-form $\overline{\Omega}$ on each stratum of $\mathcal{M}_k^0(z)$, whose lift to each of the manifolds which make up $\tilde{K}_g^{-1}(z)$ is Ω' restricted to that manifold.*

Proof. As proved in Section 3 in all the separate cases, the quotient map $\tilde{K}_g^{-1}(z) \to \tilde{K}_g^{-1}(z)/G$ is a fiber bundle projection over each $\mathcal{M}_k^0(z)$. Thus Ω' can be pulled down by smooth local sections. The properties of Ω' listed in Proposition 5.1(i) and (ii) imply that if s_1 and s_2 are two smooth local sections of $\tilde{K}_g^{-1}(z) \to \tilde{K}_g^{-1}(z)/G$ in a neighborhood of some point in $\mathcal{M}_k^0(z)$ then $s_1^*\Omega' = s_2^*\Omega'$. Thus we can define $\overline{\Omega}$ unambiguously as the 2-form, on each $\mathcal{M}_k^0(z)$, given locally by pullbacks of Ω' by smooth local sections of $\tilde{K}_g^{-1}(z) \to \tilde{K}_g^{-1}(z)/G$. Since $\mathrm{d}\Omega' = 0$ on $\tilde{K}_g^{-1}(z)$ and the fiber-bundle projection map is a submersion, it follows that $\mathrm{d}\overline{\Omega} = 0$. □

6. The symplectic structure on the $SU(2)$ moduli spaces $\mathcal{M}_k^0$

In this section we shall work with the moduli space of flat $SU(2)$ connections. The group $SU(2)$ is equipped with a fixed Ad-invariant metric $\langle \cdot, \cdot \rangle$. We will show that $\overline{\Omega}$ is a symplectic structure on $\mathcal{M}_{2g}^0$ and we will determine the corresponding symplectic volumes.

It has been proven in several works ([5], for instance) that $\overline{\Omega}$ is symplectic on $\mathcal{M}^0_{3(2g-2)}$ and the volume $\mathrm{vol}_{\overline{\Omega}}(\mathcal{M}^0_{3(2g-2)})$ has also been determined in a variety of ways [3,9].

Let T be a maximal torus in $SU(2)$, and $n \in N(T)\backslash T$, where $N(T)$ is the normalizer of T in $SU(2)$. The two-element group $W = \{I, n\}$ acts freely on $T^{2g}\backslash\{\pm I\}^{2g}$ by conjugation. Let $\mathcal{F}_{2g}$ be the subset of $K_g^{-1}(I) \subset SU(2)^{2g}$ consisting of all points where the isotropy group of the conjugation action of $SU(2)$ is a maximal torus in $SU(2)$. By definition, $\mathcal{M}^0_{2g} = \mathcal{F}_{2g}/SU(2)$. Recall from (2.10c) that the inclusion map $T^{2g}\backslash\{\pm I\}^{2g} \subset \mathcal{F}_{2g}$ induces a diffeomorphism $\overline{\overline{\Phi}} : T^{2g}\backslash\{\pm I\}^{2g}/W \to \mathcal{F}_{2g}/SU(2) = \mathcal{M}^0_{2g}$. Thus $\overline{\Omega}$ on $\mathcal{M}^0_{2g}$ is simply the projection on $T^{2g}\backslash\{\pm I\}^{2g}/W$ of the restriction $\Omega'|T^{2g}\backslash\{\pm I\}^{2g}$.

$$
\begin{array}{ccccc}
T^{2g}\backslash\{\pm I\}^{2g} & \overset{\text{inclusion}}{\longrightarrow} & \mathcal{F}_{2g} & \subset SU(2)^{2g} & \\
\downarrow & & \downarrow & & \qquad (6.1a)\\
T^{2g}\backslash\{\pm I\}^{2g}/W & \overset{\overline{\overline{\Phi}}}{\longrightarrow} & \mathcal{F}_{2g}/SU(2) & = \mathcal{M}^0_{2g} &
\end{array}
$$

Recall that we are working with a fixed Ad-invariant metric $\langle\cdot,\cdot\rangle$ on the Lie algebra of $SU(2)$, and the symplectic form $\overline{\Omega}$ is defined in terms of this metric.

Proposition 6.1.

(i) *The restriction of Ω' to T^{2g} is given on vectors $H^{(1)}, H^{(2)} \in T_xT^{2g}$ by*

$$\Omega'(H^{(1)}, H^{(2)}) = \sum_{i=1}^{g}(\langle A_i^{(1)}, B_i^{(2)}\rangle - \langle A_i^{(2)}, B_i^{(1)}\rangle), \tag{6.1b}$$

where $H^{(1)} = x \cdot (A_1^{(1)}, B_1^{(1)}, \ldots, A_g^{(1)}, B_g^{(1)})$, and $H^{(2)}$ is related similarly to the $A_i^{(2)}$ and $B_i^{(2)}$.

(ii) *The 2-form $\overline{\Omega}$ on C^0_{2g} is a symplectic form.*

(iii) *The volume of $\mathcal{M}^0_{2g}$ with respect to the symplectic form $\overline{\Omega}$ is*

$$\mathrm{vol}_{\overline{\Omega}}(\mathcal{M}^0_{2g}) = \tfrac{1}{2}[4\pi\,\mathrm{vol}(SU(2))]^{2g/3}, \tag{6.1c}$$

where $\mathrm{vol}(SU(2))$ *is the volume of* $SU(2)$ *with repect to the metric* $\langle\cdot,\cdot\rangle$.

Proof. Since each component of x is in T, it follows that, in the notation of Eq. (6.1b), $f_{i-1}^{-1}(X) = X$ for every $i \in \{1, \ldots, 4g\}$ and every $X \in \underline{t}$, the Lie algebra of T. Moreover, in (5.2a), W and Z have the form $(A_1^{(i)}, B_1^{(i)}, -A_1^{(i)}, -B_1^{(i)}, \ldots, A_g^{(i)}, B_g^{(i)}, -A_g^{(i)}, -B_g^{(i)})$. Using this in (5.2a) we see that the term involving $A_i^{(1)}$ is :

$$\tfrac{1}{2}\langle A_i^{(1)}, 0 + B_i^{(2)} - A_i^{(2)} - B_i^{(2)} + 0\rangle$$
$$+\tfrac{1}{2}\langle -A_i^{(1)}, 0 - A_i^{(2)} - B_i^{(2)} - B_i^{(2)} + 0\rangle = \langle A_i^{(1)}, B_i^{(2)}\rangle.$$

Similarly, the term involving $B_i^{(1)}$ in Eq. (5.2a) equals $-\langle B_i^{(1)}, A_i^{(2)}\rangle$. Adding up over $i = 1, \ldots, g$ yields Eq. (6.1b).

We can see directly from (6.1b) that $\Omega'|T^{2g}$ is invariant under W and thus induces a 2-form $\overline{\Omega}$ on the quotient $\simeq \mathcal{M}_{2g}^0$. Moreover, the 2-form $\Omega'|T^{2g}$ given in (6.1b), being a left invariant form on the abelian group T^{2g}, is closed; expression (6.1b) also shows that it is non-degenerate. Since the quotient map $(T^{2g}\backslash\{\pm I\}) \to \mathcal{M}_{2g}^0$ is a local diffeomorphism, we conclude that $\overline{\Omega}$ is also a symplectic form.

From (6.1b) we see that the matrix for $\Omega'|T^{2g}$ relative to a suitable orthonormal basis has block-diagonal form, with each block being

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix};$$

thus $|\det(\Omega'|T^{2g})| = 1$, and so

$$\mathrm{vol}_{\Omega'|T^{2g}}(T^{2g}\backslash\{\pm I\}^{2g}) = \mathrm{vol}_{\Omega'|T^{2g}}(T^{2g}) = \mathrm{vol}(T)^{2g},$$

where the last term is the Riemannian volume (=length) of T. Now $SU(2)$, being a 3-sphere has volume $= 2\pi^2(\text{radius})^3$, while T, being a great circle in this sphere, has length $2\pi(\text{radius})$. Thus

$$\mathrm{vol}(T) = 2\pi\left[\frac{\mathrm{vol}(SU(2))}{2\pi^2}\right]^{1/3} = [4\pi\,\mathrm{vol}(SU(2))]^{1/3}, \tag{6.1d}$$

and so

$$\mathrm{vol}_{\Omega'|T^{2g}}(T^{2g}\backslash\{\pm I\}^{2g}) = [4\pi\,\mathrm{vol}(SU(2))]^{2g/3},$$

Since $T^{2g}\backslash\{\pm I\} \to \mathcal{M}_{2g}^0$ is a two-fold cover, we have the result (6.1c). □

7. The symplectic structure on the $SO(3)$ moduli spaces $\mathcal{M}_k^0(z)$

The determination of the symplectic volumes of the different strata $\mathcal{M}_k^0(z)$ will require different methods.

7.1. $\overline{\Omega}$ on $\mathcal{M}_{2g}^0(I)$

The stratum $\mathcal{M}_{2g}^0(I)$ can be understood in a way very similar to $\mathcal{M}_{2g}^0$.

Let T be a maximal torus in $SU(2)$, and $\overline{T}$ its projection on $SO(3)$. Let $n \in N(\overline{T})\backslash\overline{T}$, where $N(\overline{T})$ is the normalizer of $\overline{T}$ in $SO(3)$. The two-element group $W = \{I, n\}$ acts freely on $\overline{T}^{2g}\backslash\{I\}^{2g}$ by conjugation. Let $\overline{\mathcal{F}}_{2g}(I)$ be the subset of $\tilde{K}_g^{-1}(I) \subset SO(3)^{2g}$ consisting of all points where the isotropy group of the conjugation action of $SO(3)$ is a maximal torus in $SO(3)$. By definition, $\mathcal{M}^0_{2g}(I) = \overline{\mathcal{F}}_{2g}(I)/SO(3)$.

Let τ be the 180° rotation belonging to $\overline{T}$. Recall, from Theorem 3.9, the commutative diagram

$$\begin{array}{ccccc} \overline{T}^{2g}\backslash\{I,\tau\}^{2g} & \stackrel{\text{inclusion}}{\longrightarrow} & \overline{\mathcal{F}}_{2g}(I) & \subset SO(3)^{2g} \\ \downarrow & & \downarrow & \\ \overline{T}^{2g}\backslash\{I,\tau\}^{2g}/W & \longrightarrow & \overline{\mathcal{F}}_{2g}(I)/SO(3) & = \mathcal{M}^0_{2g}(I) \end{array} \tag{7.1a}$$

where the lower horizontal arrow is a diffeomorphism.

Thus $\overline{\Omega}$ on $\mathcal{M}^0_{2g}(I)$ is, via the lower horizontal arrow in (7.1a), identifiable as the projection on $\overline{T}^{2g}\backslash\{I,\tau\}^{2g}/W$ of the restriction of Ω' to $\overline{T}^{2g}\backslash\{I,\tau\}^{2g}$ (the projection $\overline{T}^{2g}\backslash\{I,\tau\}^{2g} \to \overline{T}^{2g}\backslash\{I,\tau\}^{2g}/W$ is a 2-fold covering).

Recall that we are working with a fixed Ad-invariant metric $\langle\cdot,\cdot\rangle$ on the Lie algebra of $SU(2)$, and the symplectic form $\overline{\Omega}$ is defined in terms of this metric.

Proposition 7.1.

(i) *The restriction of Ω' to $\overline{T}^{2g}$ is given on vectors $H^{(1)}, H^{(2)} \in T_x\overline{T}^{2g}$ by*

$$\Omega'(H^{(1)}, H^{(2)}) = \sum_{i=1}^{g}(\langle A_i^{(1)}, B_i^{(2)}\rangle - \langle A_i^{(2)}, B_i^{(1)}\rangle), \tag{7.1b}$$

where $H^{(1)} = x \cdot (A_1^{(1)}, B_1^{(1)}, \ldots, A_g^{(1)}, B_g^{(1)})$, and $H^{(2)}$ is related similarly to the $A_i^{(2)}$ and $B_i^{(2)}$.

(ii) *The 2-form $\overline{\Omega}$ on $\mathcal{M}^0_{2g}(I)$ is a symplectic form.*

(iii) *The volume of $\mathcal{M}^0_{2g}(I)$ with respect to the symplectic form $\overline{\Omega}$ is*

$$\mathrm{vol}_{\overline{\Omega}}(\mathcal{M}^0_{2g}(I)) = \frac{1}{2}\left[\frac{\pi}{2}\mathrm{vol}(SU(2))\right]^{2g/3}, \tag{7.1c}$$

where $\mathrm{vol}(SU(2))$ is the volume of $SU(2)$ with repect to the metric $\langle\cdot,\cdot\rangle$.

Proof. The argument is virtually the same as in Proposition 6.1. For (iii), we need to observe, in addition, that

$$\mathrm{vol}_{\Omega'|\overline{T}^{2g}}(\overline{T}^{2g}\backslash\{\pm I\}^{2g}) = \mathrm{vol}_{\Omega'|\overline{T}^{2g}}(\overline{T}^{2g}) = \mathrm{vol}(\overline{T})^{2g} = \frac{1}{2^{2g}}\mathrm{vol}(T)^{2g},$$

where the last equality follows from the fact that $SU(2) \to SO(3)$ is a 2-fold covering and a local isometry. □

7.2. $\overline{\Omega}$ on $\mathcal{M}^0_{2g-2}(z)$

Recall that $\mathcal{M}^0_{2g-2}(z) \simeq (\tilde{K}_g^{-1}(z)\cap F)/SO(3)$, where F is the subset of $SO(3)^{2g}$ consisting of points where the isotropy group of the $SO(3)$-conjugation action is a two-element group. Let $\overline{T}$ be a maximal torus in $SO(3)$, $N(\overline{T})$ its normalizer, and B the subset of $N(\overline{T})^{2g}$ where the isotropy group is not a two-element group (described in detail in (3.11c), and (3.11d). We have the commutative diagram

$$\begin{array}{ccc} \tilde{K}_g^{-1}(z) \cap (N(\overline{T})^{2g} \backslash B) & \overset{\text{inclusion}}{\rightarrow} & \tilde{K}_g^{-1}(z) \cap F \\ \downarrow p & & \downarrow p' \\ [\tilde{K}_g^{-1}(z) \cap (N(\overline{T})^{2g} \backslash B)]/N(\overline{T}) & \overset{\overline{\overline{\Psi}}}{\rightarrow} & (\tilde{K}_g^{-1}(z) \cap F)/SO(3) \simeq \mathcal{M}^0_{2g-2}(z) \end{array} \tag{7.2a}$$

where the bottom arrow is a diffeomorphism.

Let $N'(\overline{T}) = N(\overline{T})/\{I, \tau\}$, where τ is the 180° rotation in $\overline{T}$. The vertical arrow on the left in (7.2a) is a fiber bundle projection, and in fact it is a principal $N'(\overline{T})$-bundle. Thus $\overline{\Omega}|\mathcal{M}^0_{2g-2}(z)$ is the 2-form induced via p by $\Omega'|\tilde{K}_g^{-1}(z) \cap (N(\overline{T})^{2g} \backslash B)$.

Since the conjugation action of $N(\overline{T})$ on $N(\overline{T})^{2g}$ is by isometries, the fiber bundle projection p induces, in a natural way, a Riemannian metric on $[\tilde{K}_g^{-1}(z) \cap (N(\overline{T})^{2g} \backslash B)]/N(\overline{T})$. We shall equip $\mathcal{M}^0_{2g-2}(z)$ with the corresponding Riemannian metric induced via $\overline{\overline{\Psi}}$. (A vector in some $T_pN(\overline{T})^{2g}$ which is perpendicular to the $N(\overline{T})$-orbit through p is automatically perpendicular to the $SO(3)$-orbit through p; thus $\overline{\overline{\Psi}}$ is an isometry when the domain and image of $\overline{\overline{\Psi}}$ are equipped with the quotient metrics).

We work with $J = \{1, 2, 5, 6, \dots, 4g-3, 4g-2\}$, as in (5.1a).

For $S \subset J$, recall from (3.16a) and (3.16b) the map $\phi_S : \overline{T}^{2g} \to N(\overline{T})^{2g}$. If $\alpha \in \phi_S(\overline{T}^{2g})$ then, by definition of ϕ_S,

$$\alpha_j \in \begin{cases} \overline{T} & \text{if } j \in S, \\ N(\overline{T}) \backslash \overline{T} & \text{if } j \in J \backslash S. \end{cases} \tag{7.2b}$$

Thus, for $\alpha \in \phi_S(\overline{T}^{2g})$,

$$\mathrm{Ad}(\alpha_j)|\underline{t} = \begin{cases} I & \text{if } j \in S, \\ -I & \text{if } j \in J \backslash S. \end{cases} \tag{7.2c}$$

where I is the identity map on $\underline{t}$.

We have the orbit map $\gamma_\alpha : N(\overline{T}) \to N(\overline{T})^{2g} : x \mapsto x\alpha x^{-1}$, whose derivative, at the identity in $\underline{t}$, is given by a linear map $\mathrm{d}\gamma_\alpha : \underline{t} \to \underline{t}^{2g}$. On the other hand, we have the product commutator map $\tilde{K}_g : \overline{T}^{2g} \to T$, whose derivative is described by a linear map $\mathrm{d}\tilde{K}_g|_\alpha : \underline{t}^{2g} \to \underline{t}$ (all tangent vectors left-translated to the identity).

Lemma 7.2. *For any $S \subset J$, and $\alpha \in \phi_S(\overline{T})^{2g}$,*

$$\det(\mathrm{d}\gamma_\alpha|\underline{t}) = 2\sqrt{2g - \#S} = \det(\mathrm{d}\tilde{K}_g|^*_\alpha \underline{t}). \tag{7.2d}$$

Proof. Differentiating the expression $\gamma_\alpha(x) = x\alpha x^{-1}$ at x equal to the identity, we have for any $X \in \underline{t}$:

$$\mathrm{d}\gamma_\alpha(X) = ([\mathrm{Ad}(\alpha_j^{-1}) - 1]X)_{j \in J}.$$

Thus, by (7.2c), the jth entry of $\mathrm{d}\gamma_\alpha(X)$ is 0 if $j \in S$ and it is $-2X$ if $j \in J\backslash S$. Thus $\det \mathrm{d}\gamma_\alpha|\underline{t} = 2\sqrt{\#(J\backslash S)} = 2\sqrt{2g - \#S}$.

Recall that we write α as $(\alpha_j)_{j\in J}$, where $J = \{1, 2, 5, 6, \ldots, 4g-3, 4g-2\}$. Then $\tilde{K}_g(\alpha) = \tilde{\alpha}_{4g}\tilde{\alpha}_{4g-1}\cdots\tilde{\alpha}_1$, where, for each $j \in J$, $\tilde{\alpha}_{j+2} = \tilde{\alpha}_j^{-1}$ and $\tilde{\alpha}_j \in T \subset SU(2)$ is any element covering α_j. Then

$$\tilde{K}_g(\alpha)^{-1}\mathrm{d}\tilde{K}_g(\alpha H) = \sum_{j\in J}(f_{j-1}^{-1} - f_{j+2}^{-1})H_j,$$

where $f_j = \mathrm{Ad}(\alpha_j\alpha_{j-1}\cdots\alpha_1)$. Taking the adjoint, we have

$$\mathrm{d}\tilde{K}_g|_\alpha^* X = ((f_{j-1} - f_{j+2})X)_{j\in J}, \tag{7.2e}$$

here we are working with $X \in \underline{t}$, in which case $\mathrm{d}\tilde{K}_g|_\alpha^* X \in \underline{t}^{2g}$ (the formulas are all valid for $\underline{g}$ in place of $\underline{t}$). Since $\mathrm{Ad}(\alpha_i)|\underline{t} = \pm I$, the different $\mathrm{Ad}(\alpha_i)|\underline{t}$'s commute, and so, for any $j \in J$:

$$\begin{aligned} f_{j+2} &= \mathrm{Ad}(\alpha_{j+2}\alpha_{j+1}\alpha_j)f_{j-1} \\ &= \mathrm{Ad}(\alpha_j^{-1}\alpha_{j+1}\alpha_j)f_{j-1} \\ &= \mathrm{Ad}(\alpha_{j+1})f_{j-1} = \begin{cases} f_{j-1} & \text{if } j+1 \in S \cup (S+2), \\ -f_{j-1} & \text{otherwise,} \end{cases} \end{aligned}$$

where in the last step we used (7.2c) and $\alpha_{j+2} = \alpha_j^{-1}$. Thus

$$j\text{th component of } d\tilde{K}_g|_\alpha^* X \text{ is} = \begin{cases} 0 & \text{if } j+1 \in S \cup (S+2), \\ 2f_{j-1}X = \pm 2X & \text{otherwise.} \end{cases}$$

Thus

$$\det(\mathrm{d}\tilde{K}_g|_\alpha^*) = 2\sqrt{2g - \#S'},$$

where $S' = \{j \in J: j+1 \in S \cup (S+2)\}$. Now the mapping $f : S' \to S : j \mapsto f(j)$, where $f(j) = j \pm 1$ according as $j \pm 1 \in S$, is a bijection, So $\#S' = \#S$, and so $\det(\mathrm{d}\tilde{K}_g|_\alpha^*)$ is as in (7.2d). □

Proposition 7.3. *The 2-form* $\overline{\Omega}|\mathcal{M}^0_{2g-2}(z)$ *is symplectic. Moreover, on* $\mathcal{M}^0_{2g-2}(z)$

$$\mathrm{Pfaffian}(\overline{\Omega}) = 1, \tag{7.3a}$$

i.e. the volume measure on $\mathcal{M}^0_{2g-2}(z)$ *induced by the symplectic form* $\overline{\Omega}$ *is the same as the Riemannian volume measure.*

Proof. It is proved in [5] that

$$\mathrm{Pfaffian}(\overline{\Omega}) = \frac{\det \mathrm{d}\gamma_\alpha|\underline{t}}{\det \mathrm{d}\tilde{K}_g|_\alpha^*\underline{t}}. \tag{7.3b}$$

(The argument in [5] is for $\underline{g}$ and $\overline{\Omega}|\mathcal{M}^0_{3(2g-2)}(z)$ but is valid without any change in the present simpler situation.) The result now follows from Lemma 7.2. □

Proposition 7.4. *The symplectic volume, with respect to the symplectic structure $\overline{\Omega}$, of each connected component of $\mathcal{M}^0_{2g-2}(z)$ is $\frac{1}{2}[\pi \operatorname{vol}(SU(2))/2]^{(2g-2)/3}$.*

Proof. Recall from Theorem 3.20 that $\mathcal{M}^0_{2g-2}(z)$ is the union of $2^{2g}-1$ connected components $\mathcal{M}^0_{2g-2}(z)_S$, one for each proper subset S of $J = \{1, 2, 5, 6, \dots, 4g-3, 4g-2\}$, and $\mathcal{M}^0_{2g-2}(z)_S \simeq ((\tilde{K}_g \circ \phi_S)^{-1}(z)\backslash B_S)/N'(\overline{T})$.

Since the symplectic volume measure $\operatorname{vol}_{\overline{\Omega}}$ coincides with the Riemannian volume measure on $\mathcal{M}^0_{2g-2}(z)$, it follows from Lemma 4.3 and the determinant values in (7.2d) that

$$\operatorname{vol}_{\overline{\Omega}}(\mathcal{M}^0_{2g-2}(z)_S) = \frac{1}{\operatorname{vol}(N'(\overline{T}))}\frac{1}{2\sqrt{2g-\#S}}\operatorname{vol}[\tilde{K}_g^{-1}(z)\cap \phi_S(\overline{T}^{2g}\backslash B_S)], \tag{7.4a}$$

where vol (with no subscript) is Riemannian volume.

Since ϕ_S is an isometry and B_S is a submanifold of positive codimension in $\overline{T}^{2g}$, it follows that the Riemannian volume appearing on the right side in (7.4a) equals the Riemannian volume of $(\tilde{K}_g \circ \phi_S)^{-1}(z)$.

Now, as observed in Proposition 3.17,

$$(\tilde{K}_g \circ \phi_S)(t_j)_{j\in J} = \prod_{j\in J} \tilde{t}_j^{m_j}, \tag{7.4b}$$

where $\tilde{t}_j$ is any element of T covering $t_j \in \overline{T}$, and, for $j = 1, 5, \dots, 4g-3$,

$$(m_j, m_{j+1}) = \begin{cases} (0,0) & \text{if } j, j+1 \in S, \\ (2,0) & \text{if } j \in S \text{ and } j+1 \notin S, \\ (0,-2) & \text{if } j \notin S \text{ and } j+1 \in S, \\ (-2,2) & \text{if } j \notin S \text{ and } j+1 \notin S. \end{cases} \tag{7.4c}$$

Fixing a $j_* \in J\backslash S$, the map $\overline{T}^{2g} \to \overline{T}^{2g-1}$ which carries $(x_j)_{j\in J}$ to the projection $(x_j)_{j\in J, j\neq j_*}$ is a bijection of $(\tilde{K}_g \circ \phi_S)^{-1}(z)$ onto $\overline{T}^{2g-1}$. The Jacobian of the inverse map $\overline{T}^{2g-1} \to (\tilde{K}_g \circ \phi_S)^{-1}(z)$ is $(1/|m_{j_*}|)\sqrt{\sum_{j\in J} m_j^2}$. The specification of the m_j above shows that this Jacobian equals $\sqrt{\#(J\backslash S)}$. So

$$\operatorname{vol}((\tilde{K}_g \circ \phi_S)^{-1}(z)) = \sqrt{2g-\#S}\ \operatorname{vol}(\overline{T}^{2g-1}). \tag{7.4d}$$

Substituting this into (7.4a), and using $\operatorname{vol}(\overline{T}) = \frac{1}{2}\operatorname{vol}(T)$, as well as the value of $\operatorname{vol}(T)$ mentioned in (6.1d) we have

$$\begin{aligned}\operatorname{vol}_{\overline{\Omega}}(\mathcal{M}^0_{2g-2,S}(z)) &= \frac{1}{\operatorname{vol}(\overline{T})}\frac{1}{2\sqrt{2g-\#S}}\sqrt{2g-\#S}\ \operatorname{vol}(\overline{T}^{2g-1}) \\ &= \frac{1}{2}\operatorname{vol}(\overline{T})^{2g-2}\end{aligned}$$

$$= \frac{1}{2}\left[\frac{1}{2}2\pi\left(\frac{\mathrm{vol}(SU(2))}{2\pi^2}\right)^{1/3}\right]^{2g-2}$$
$$= \frac{1}{2}\left[\frac{\pi}{2}\mathrm{vol}(SU(2))\right]^{(2g-2)/3} \qquad \square$$

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